Surgery formula for torsions and Seiberg-Witten invariants of 3-manifolds

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Abstract

We give a surgery formula for the torsions and Seiberg-Witten invariants associated with $Spin^c$-structures on 3-manifolds. We use the technique of Reidemeister-type torsions and their refinements.

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Introduction

In 1976 the author introduced a combinatorial torsion-type invariant, τ, of compact PL-manifolds of any dimension (see [Tu2]). For a compact 3-manifold M, the invariant $\tau(M)$ lies in the group ring $\mathbf{Z}[H_1(M)]$ if $b_1(M) \geq 2$ and lies in a certain extension of this ring if $b_1(M) = 0, 1$. The study of $\tau(M)$ was motivated by its connections with the Alexander-Fox invariants of M including the Alexander polynomial $\Delta(M)$.

The definition of τ contains an indeterminacy so that $\tau(M)$ is defined only up to multiplication by ± 1 and by elements of $H_1(M)$. In [Tu3], [Tu4] the author introduced a refined version $\tau(M, e, \omega)$ of $\tau(M)$ depending on the choice of a so-

called Euler structure e on M and a homology orientation ω of M. The invariant $\tau(M,e,\omega)$ has no indeterminacy and $\tau(M)=\pm H_1(M)\,\tau(M,e,\omega)$ for all e,ω.

The Seiberg-Witten invariant of a closed oriented 3-manifold M with $b_1(M)\geq 1$ is a numerical function, $SW=SW(M)$, on the set of $Spin^c$-structures on M, see for instance [FS], [HL], [Li], [MT], [Mo], [MOY], [OT]. This function also depends on the choice of a homology orientation, ω, of M. For a $Spin^c$-structure e on M, the integer $SW(e,\omega)$ is the algebraic number of solutions, called monopoles, to a certain system of differential equations associated with e. This number coincides with the 4-dimensional SW-invariant of the $Spin^c$-structure $e\times 1$ on $M\times S^1$.

The invariants $SW(M)$ and $\tau(M)$ turn out to be equivalent (at least up to sign). The first step in this direction was made by Meng and Taubes [MT] (see also [FS]) who observed that $SW(M)$ determines the Alexander polynomial $\Delta(M)$. The equivalence between $SW(M)$ and $\tau(M)$ was established in [Tu5], [Tu6] where the SW-invariants of $Spin^c$-structures on M are identified with the coefficients in the expansion of $\tau(M)$ as an element of the group ring. This involves an identification of the Euler structures on M with the $Spin^c$-structures on M.

The definition of τ is based on the methods of the theory of torsions, specifically, triangulations, chain complexes, etc. The definition of SW is analytical. These definitions are not always suitable for explicit computations. The aim of this paper is to give surgery formulas for τ and SW suitable for computations.

We first give a surgery description of Euler structures (= $Spin^c$-structures) on closed oriented 3-manifolds. To this end we introduce a notion of a charge. A *charge* on an oriented link $L=L_1\cup\ldots\cup L_m$ in S^3 is an m-tuple $(k_1,\ldots,k_m)\in\mathbf{Z}^m$ such that for all $i=1,\ldots,m$,

$$k_i\equiv 1+\sum_{j\neq i} lk(L_i,L_j) (\operatorname{mod} 2) \qquad (0.a)$$

where lk is the linking number in S^3. We show that a charge k on L determines an Euler structure, e_k^M, on any 3-manifold M obtained by surgery on L. Similarly, the orientation of L induces a homology orientation, ω_L^M, of M.

Our main formula computes $\tau(M,e_k^M,\omega_L^M)$ in terms of the framing and linking numbers of the components of L, and the Alexander-Conway polynomials of L and its sublinks. This implies a surgery formula for $\pm SW(e_k^M,\omega_L^M)$. Thus, the algebraic number of monopoles can be computed (at least up to sign) in terms of classical link invariants.

For the sake of introduction, we state our surgery formula for SW in the case of 3-manifolds with $b_1\geq 2$ obtained by surgery on algebraically split links. Recall that the Alexander-Conway polynomial ∇_L of an oriented link $L=L_1\cup\ldots\cup L_m\subset S^3$ with $m\geq 2$ is a Laurent polynomial, i.e., an element of $\mathbf{Z}[t_1^{\pm1},\ldots,t_m^{\pm1}]$. If L is algebraically split, i.e., if $lk(L_i,L_j)=0$ for all $i\neq j$, then ∇_L is divisible by $\prod_{i=1}^m(t_i^2-1)$ in $\mathbf{Z}[t_1^{\pm1},\ldots,t_m^{\pm1}]$. We have a finite expansion

$$\nabla_L/\prod_{i=1}^m(t_i^2-1)=\sum_{l=(l_1,\ldots,l_m)\in\mathbf{Z}^m} z_l(L)\ t_1^{l_1}\ldots t_m^{l_m}$$

with $z_l(L) \in \mathbf{Z}$.

Let M be a closed oriented 3-manifold with $b_1(M) \geq 2$ obtained by surgery on a framed oriented algebraically split link $L = L_1 \cup ... \cup L_m \subset S^3$. Let $f = (f_1, ..., f_m)$ be the tuple of the framing numbers of $L_1, ..., L_m$. Denote by J_0 the set of all $j \in \{1, ..., m\}$ such that $f_j = 0$. For a set $J \subset \{1, ..., m\}$ we denote the link $\cup_{j \in J} L_j$ by L^J. Put $\overline{J} = \{1, ..., m\} \backslash J$ and $|J| = \mathrm{card}(J)$. Then for any charge $k = (k_1, ..., k_m)$ on L,

$$\pm\, SW(e_k^M, \omega_L^M) \qquad (0.b)$$

$$= \sum_{J_0 \subset J \subset \{1,...,m\}} (-1)^{|J|} \prod_{j \in \overline{J}} \mathrm{sign}(f_j) \sum_{l \in \mathbf{Z}^J, l \equiv -k (\mathrm{mod}\, 2f)} z_l(L^J).$$

Here the sum goes over all sets $J \subset \{1, ..., m\}$ containing J_0. The sign $\mathrm{sign}(f_j) = \pm 1$ of f_j is well defined since $f_j \neq 0$ for $j \in \overline{J}$. The formula $l \in \mathbf{Z}^J, l \equiv -k(\mathrm{mod}\, 2f)$ means that l runs over all tuples of integers numerated by elements of J such that $l_j \equiv -k_j(\mathrm{mod}\, 2f_j)$ for all $j \in J$. By $|J| \geq |J_0| = b_1(M) \geq 2$, the algebraically split link L^J has ≥ 2 components so that $z_l(L^J)$ is a well defined integer. Only a finite number of these integers are non-zero and therefore the sum on the right-hand side of (0.b) is finite. This sum obviously depends only on $k(\mathrm{mod}\, 2f)$; the Euler structure e_k^M also depends only on $k(\mathrm{mod}\, 2f)$. The precise sign in front of SW in (0.b) is unknown to the author. This sign is the same for all charges k on L.

For a link $L = L_1 \cup ... \cup L_m$ which is not algebraically split, the polynomial ∇_L can be divided by $\prod_{i=1}^m (t_i^2 - 1)$ in a certain quotient of $\mathbf{Z}[t_1^{\pm 1}, ..., t_m^{\pm 1}]$. This leads to a general surgery formula for an arbitrary L.

We give also a surgery formula for the Alexander polynomial $\Delta(M)$. It is simpler than the surgery formulas for τ and SW, and we establish it first.

The methods of this paper yield similar surgery formulas for the invariants τ, Δ, SW of link exteriors in closed oriented 3-manifolds.

The invariants τ, Δ, SW are closely related to the Casson-Walker-Lescop invariant of 3-manifolds. We shall discuss these relations and other properties of τ, Δ, SW in another place.

The organisation of the paper should be clear from Contents above.

1. Smooth Euler structures on 3-manifolds

1.1. Euler structures on manifolds. We recall briefly the theory of (smooth) Euler structures on manifolds following [Tu4]. By a *relative vector field* on a compact (smooth) manifold M we mean a *nonsingular* tangent vector field on M directed outside M on ∂M (transversely to ∂M). Relative vector fields u and v on M are called *homologous* if for every connected component M_0 of M and a point $x \in M_0$ the restrictions of u and v to $M_0 \backslash \{x\}$ are homotopic in the class of nonsingular vector fields on $M_0 \backslash \{x\}$ directed outside M_0 on ∂M_0. The homology class of a relative vector field u on M is called a (smooth) *Euler structure* on M and denoted by $[u]$. The set of Euler structures on M is denoted by $\mathrm{vect}(M)$. This set is non-void iff for each connected component M_0 of M we have $\chi(M_0) = 0$. This

condition is satisfied for instance if M is a compact 3-manifold whose boundary consists of tori.

The group $H_1(M)$ acts on $\mathrm{vect}(M)$ as follows. (This action as well as the group operation in $H_1(M)$ will be written multiplicatively.) Let u be a relative vector field on M. Let $h \in H_1(M)$ be represented by an oriented simple closed curve $l \subset M \backslash \partial M$. Let V be a regular neighborhood of l. We endow V with coordinates (θ, r), where $\theta \in \mathbf{R}/2\pi\mathbf{Z}$ is the coordinate along l and r is the radial coordinate on the discs transversal to l. Choose a Riemannian metric on V such that the tangent vector fields $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial r}$ are orthogonal everywhere and the Euclidean norm $|r| \geq 0$ measures the distance of a point of V from l. We can assume that $|r|_{|\partial V} = 1$, and after applying a homotopy to u we can assume that $u_{|V} = -\frac{\partial}{\partial\theta}$. Then $h[u]$ is represented by the relative vector field on M which is equal to u on $M \setminus V$ and is equal to $\cos(|r|\pi)\,\frac{\partial}{\partial\theta} + \sin(|r|\pi)\,\frac{\partial}{\partial r}$ on V.

If $\mathrm{vect}(M) \neq \emptyset$ then the action of $H_1(M)$ on $\mathrm{vect}(M)$ is free and transitive. Thus, for any two relative vector fields u, v on M, there is a unique $h \in H_1(M)$ such that $[u] = h[v]$. This h is denoted by $[u]/[v]$. Clearly, $[v]/[u] = ([u]/[v])^{-1}$.

For the torus $T = S^1 \times S^1$, the set $\mathrm{vect}(T)$ has a distinguished element defined as follows. Consider a non-singular vector field w on T whose flow lines are the circles $x \times S^1$ with $x \in S^1$. The class $[w] \in \mathrm{vect}(T)$ depends neither on the orientation of the circles nor on the choice of the splitting of T as a product of two circles. This follows from the easy observation that $[w]$ is invariant under the Dehn twists along $S^1 \times x$ and $x \times S^1$. In fact $[w]$ is the only element of $\mathrm{vect}(T)$ invariant under all self-homeomorphisms of T. The formula $h \mapsto h[w]$ defines a natural bijection $H_1(T) \to \mathrm{vect}(T)$. We call a non-singular vector field on T *canonical* if it is homologous to w.

1.2. The Chern class. Let M be a compact 3-manifold whose boundary consists of tori. For any $u \in \mathrm{vect}(M)$, the Chern class $c(u) \in H_1(M)$ can be defined as follows: consider a non-singular tangent vector field on ∂M whose restrictions to all components are canonical; the first obstruction to extending this vector field to M transversely to u lies in $H^2(M, \partial M)$ and $c(u) \in H_1(M)$ is its Poincaré dual. We however shall use a somewhat different definition of $c(u)$.

For a relative vector field u on M, we define an opposite vector field as follows. Deforming if necessary u in a regular neighborhood $\partial M \times [0,1] \subset M$ of $\partial M = \partial M \times 0$ we can assume that u is tangent to the lines $x \times [0,1]$ for $x \in \partial M$ and directed from 1 to 0. Consider the vector field on M which is equal to $-u$ on the complement of $\partial M \times [0,1]$ in M and is given by the formula $(x,t) \mapsto \cos(t\pi)u(x,t) + \sin(t\pi)w(x)$ on $\partial M \times [0,1]$ where $x \in \partial M, t \in [0,1]$ and w is the canonical vector field on the tori forming ∂M. It is clear that this is a relative vector field on M. Its class in $\mathrm{vect}(M)$ depends only on $e = [u]$. We denote this class by e^{-1}. It follows from definitions that $(he)^{-1} = h^{-1}e^{-1}$ for any $h \in H_1(M), e \in \mathrm{vect}(M)$.

For $e \in \mathrm{vect}(M)$, set $c(e) = e/e^{-1}$. We have the identities

$$c(he) = (he)/(h^{-1}e^{-1}) = h^2(e/e^{-1}) = h^2 c(e).$$

This formula has useful applications. It implies that the negation is an involution

on $\mathrm{vect}(M)$. Indeed, we have

$$c(e^{-1}) = c((e^{-1}/e)e) = (e^{-1}/e)^2\, c(e) = c(e)^{-2}c(e) = c(e)^{-1} \tag{1.2.a}$$

and therefore

$$(e^{-1})^{-1} = (c(e^{-1}))^{-1}e^{-1} = (c(e)^{-1})^{-1}e^{-1} = c(e)e^{-1} = e.$$

The formula $c(he) = h^2c(e)$ shows that the Chern class map $e \mapsto c(e) : \mathrm{vect}(M) \to H_1(M)$ is injective provided $H_1(M)$ has no 2-torsion. The formula $c(he) = h^2c(e)$ implies also that the class $c(e) (\mathrm{mod}\, 2) \in H_1(M; \mathbf{Z}/2\mathbf{Z})$ does not depend on e. This class can be computed as follows.

1.3. Lemma. *Let M be a compact orientable 3-manifold whose boundary consists of tori. Let $\Sigma \subset M$ be a compact embedded surface such that $\partial\Sigma = \partial M \cap \Sigma$ and all components of $\partial\Sigma$ are non-contractible on ∂M. Then for any $e \in \mathrm{vect}(M)$ we have*

$$c(e) \cdot \Sigma = b_0(\partial\Sigma) (\mathrm{mod}\, 2)$$

where $c(e) \cdot \Sigma$ is the intersection number of $c(e)$ with Σ modulo 2 and $b_0(\partial\Sigma)$ is the number of components of $\partial\Sigma$.

This lemma completely determines $c(e)(\mathrm{mod}\, 2)$ since the homology classes of surfaces Σ as in the lemma generate the group $H_2(M, \partial M; \mathbf{Z}/2\mathbf{Z})$ dual to $H_1(M; \mathbf{Z}/2\mathbf{Z})$. We give a proof of this lemma at the end of Sect. 1.

1.4. Example. A *solid torus* is the product $S^1 \times D^2$ where S^1 is a circle and D^2 is a closed 2-disc. A solid torus $\mathrm{Z} = S^1 \times D^2$ endowed with a generator, h_{Z}, of $H_1(\mathrm{Z}) = \mathbf{Z}$ is said to be *directed*. In other words, Z is directed if its core circle $S^1 \times pt$ with $pt \in D^2$ is oriented. Since $H_1(\mathrm{Z})$ has no torsion, the Chern class map $\mathrm{vect}(\mathrm{Z}) \to H_1(\mathrm{Z})$ is injective. Applying Lemma 1.3 to the meridional disc $\Sigma = x \times D^2 \subset \mathrm{Z}$ with $x \in S^1$ we obtain that the image of this map consists of the odd powers of the generator. Thus, on a directed solid torus Z there is a unique Euler structure e such that $c(e) = h_{\mathrm{Z}}^{-1}$. We call this $e \in \mathrm{vect}(\mathrm{Z})$ the *distinguished Euler structure* on Z and denote it by e_{Z}.

1.5. Gluing of Euler structures. Let M be a compact 3-manifold whose boundary consists of tori. Let $T \subset M \backslash \partial M$ be a finite system of disjoint embedded 2-tori splitting M into two 3-manifolds M_0 and M_1. We define now a gluing map

$$\cup : \mathrm{vect}(M_0) \times \mathrm{vect}(M_1) \to \mathrm{vect}(M). \tag{1.5.a}$$

Let u_i be a relative vector field on M_i for $i = 0, 1$. We can identify a regular neighborhood of T in M with $T \times [0, 1]$ so that

$$T = T \times (1/2), \quad T \times [0, 1/2] \subset M_0, \quad T \times [1/2, 1] \subset M_1.$$

We deform u_0 so that it is tangent to the lines $x\times[0,1/2]$ for $x\in T$ and is directed from 0 to 1/2. Similarly, we deform u_1 so that it is tangent to the lines $x\times[1/2,1]$ for $x\in T$ and is directed from 1 to 1/2. Let w be a non-singular tangent vector field on T whose restriction to each component of T is canonical in the sense of Sect. 1.1. We define a relative vector field $u_0\cup u_1$ on M as follows: on $M_0\backslash(T\times[0,1/2])$ it is equal to u_0, on $M_1\backslash(T\times[1/2,1])$ it is equal to u_1, at any point $(x,t)\in T\times[0,1/2]$ it equals $\cos(t\pi)u_0(x,t)+\sin(t\pi)w(x)$, at any point $(x,t)\in T\times[1/2,1]$ it equals $-\cos(t\pi)u_1(x,t)+\sin(t\pi)w(x)$. The class $[u_0\cup u_1]\in \mathrm{vect}(M)$ depends only on $[u_0],[u_1]$. This yields a well-defined map (1.5.a).

It follows from definitions that for any $h_i\in H_1(M_i), e_i\in\mathrm{vect}(M_i)$ with $i=0,1$,

$$h_0e_0\cup h_1e_1=(\mathrm{in}_0(h_0)\mathrm{in}_1(h_1))\,(e_0\cup e_1) \tag{1.5.b}$$

where in_i is the inclusion homomorphism $H_1(M_i)\to H_1(M)$. As an exercise, the reader may check that

$$(e_0\cup e_1)^{-1}=e_0^{-1}\cup e_1^{-1}$$

(hint: use that w is homotopic to $-w$). This implies that

$$c(e_0\cup e_1)=(e_0\cup e_1)/(e_0\cup e_1)^{-1}$$

$$=(c(e_0)e_0^{-1}\cup c(e_1)e_1^{-1})/(e_0^{-1}\cup e_1^{-1})=\mathrm{in}_0(c(e_0))\,\mathrm{in}_1(c(e_1)).$$

We shall use the gluing of Euler structures in the following setting. Consider a 3-manifold M obtained from a compact 3-manifold E by gluing n directed solid tori $\mathrm{Z}_1,...,\mathrm{Z}_n$ along n tori in ∂E. Gluing to $e\in\mathrm{vect}(M)$ the distinguished Euler structures on these solid tori we obtain an Euler structure, e^M, on M:

$$e^M=e\cup e_{\mathrm{Z}_1}\cup...\cup e_{\mathrm{Z}_n}\in\mathrm{vect}(M).$$

Clearly,

$$c(e^M)=\mathrm{in}(c(e))\prod_{j=1}^n h_j^{-1} \tag{1.5.c}$$

where $\mathrm{in}: H_1(E)\to H_1(M)$ is the inclusion homomorphism and $h_1,...,h_n\in H_1(M)$ are the homology classes of the oriented core circles of $\mathrm{Z}_1,...,\mathrm{Z}_n$. We have $(e^M)^{-1}=\prod_{j=1}^n h_j(e^{-1})^M$ and $(he)^M=\mathrm{in}(h)\,e^M$ for any $h\in H_1(E)$.

1.6. Charges on links and Euler structures on link exteriors. The Euler structures on link exteriors can be described in terms of charges. Let L be an oriented link in an oriented 3-dimensional integral homology sphere N. A *charge* on L is an integer-valued function k on the set of components of L such that for every component ℓ of L we have $k(\ell)\equiv 1+lk(\ell,L\backslash\ell)(\mathrm{mod}\,2)$ where lk denotes the linking number in N.

Consider the exterior E of L in N. There is a canonical bijection between the set $\mathrm{vect}(E)$ and the set of charges on L. It is defined as follows. For a component

ℓ of L denote by $t_\ell \in H_1(E)$ the homology class of a meridian of ℓ. (By a meridian of an oriented knot we always mean a *canonically oriented* meridian whose linking number with the knot equals $+1$). For $e \in \mathrm{vect}(E)$, the Chern class $c(e) \in H_1(E)$ can be uniquely expanded as $\prod_\ell t_\ell^{k(\ell)}$ where ℓ runs over the components of L and $k(\ell) \in \mathbf{Z}$. We claim that (i) the function $\ell \mapsto k(\ell)$ is a charge on L and (ii) the resulting map from $\mathrm{vect}(E)$ to the set of charges on L is bijective. To prove (i), consider a component ℓ of L and its Seifert surface $\Sigma_\ell \subset N$ intersecting $L\backslash\ell$ transversely. Then the surface $\Sigma = \Sigma_\ell \cap E \subset E$ satisfies the conditions of Lemma 1.3. Note that $t_\ell \cdot \Sigma = 1 (\mathrm{mod}\, 2)$ and $t_{\ell'} \cdot \Sigma = 0$ for any component ℓ' of $L\backslash\ell$. This and Lemma 1.3 imply that

$$k(\ell) \equiv c(e) \cdot \Sigma \equiv b_0(\partial\Sigma) \equiv 1 + lk(\ell, L\backslash\ell) (\mathrm{mod}\, 2).$$

Claim (ii) is essentially obvious: the injectivity follows from the injectivity of the Chern class map $c : \mathrm{vect}(E) \to H_1(E)$ and the surjectivity follows from the identity $c(he) = h^2 c(e)$ where $e \in \mathrm{vect}(E)$, $h \in H_1(E)$.

For a charge k on L, denote by e_k the unique Euler structure on E such that $c(e_k) = \prod_\ell t_\ell^{k(\ell)}$. Formula (1.2.a) implies that $(e_k)^{-1} = e_{-k}$ where $-k$ is the charge on L defined by $(-k)(\ell) = -k(\ell)$. For any charges k, k' on L,

$$e_{k'} = \prod_\ell t_\ell^{(k'(\ell)-k(\ell))/2}\, e_k.$$

If $L = L_1 \cup ... \cup L_m$ is ordered then a charge on L is just an m-tuple $(k_1, ..., k_m) \in \mathbf{Z}^m$ satisfying formula (0.a) for all $i = 1, ..., m$. For a set $I \subset \{1, ..., m\}$, we denote by L^I the sublink $\{L_i\}_{i \in I}$ of L. A charge k on L induces a charge k^I on L^I by

$$k^I(\ell) = k(\ell) - lk(\ell, L\backslash L^I) = k(\ell) - lk(\ell, L^{\overline{I}})$$

where ℓ runs over the components of L^I and $\overline{I} = \{1, ..., m\}\backslash I$. The corresponding map $\mathrm{vect}(E) \to \mathrm{vect}(E^I)$ (where E^I is the exterior of L^I) is described as follows. We can obtain E^I from E by gluing regular neighborhoods of the components of $L^{\overline{I}}$. These regular neighborhoods are directed solid tori: the orientation of their cores is determined by the orientation of L. Gluing to $e_k \in \mathrm{vect}(E)$ the distinguished Euler structures on these directed solid tori we obtain $e_{k^I} \in \mathrm{vect}(E^I)$, so that in the notation of Sect. 1.5 we have $e_{k^I} = (e_k)^{E^I}$. This follows from (1.5.c) applied to the inclusion $E \subset E^I$. (Here $h_1, ..., h_n$ are the homology classes of the components of $L^{\overline{I}}$ in E^I.)

1.7. Surgery presentation of Euler structures. Let M be a closed 3-manifold obtained by Dehn surgery along an oriented link $L = L_1 \cup ... \cup L_m$ in an oriented 3-dimensional integral homology sphere N. This means that M is obtained by gluing m directed solid tori $\mathrm{Z}_1, ..., \mathrm{Z}_m$ to the exterior E of L. Consider a charge k on L. Gluing to $e_k \in \mathrm{vect}(E)$ the distinguished Euler structures on $\mathrm{Z}_1, ... \mathrm{Z}_m$ we

obtain an Euler structure $e_k^M = (e_k)^M$ on M. If $k, k' \in \mathbf{Z}^m$ are two charges on L then $k' - k = 2h$ with $h \in \mathbf{Z}^m = H_1(E)$ and $e_{k'} = he_k, e_{k'}^M = \mathrm{in}(h)e_k^M$ where $\mathrm{in} : H_1(E) \to H_1(M)$ is the inclusion homomorphism. This implies that $e_{k'}^M = e_k^M$ if and only if $h = (k' - k)/2 \in \mathrm{Ker}(\mathrm{in})$. Since $\mathrm{in} : H_1(E) \to H_1(M)$ is surjective we obtain that all Euler structures on M have the form e_k^M for a charge k on M.

For an integral surgery on L determined by a framing of L, we shall always use the following structure of a directed solid tori on $\mathrm{Z}_1, ..., \mathrm{Z}_m$. Let $U_1, ..., U_m$ be disjoint closed regular neighborhoods of $L_1, ..., L_m$ so that $E = N \setminus \cup_{i=1}^m \mathrm{Int}(U_i)$. Let Z_i be the solid tori glued to $\partial U_i \subset \partial E$ to form M where $i = 1, ..., m$. A meridian of L_i lying in ∂U_i is a core of Z_i. The canonical orientation of this meridian induced by the orientation of L_i makes Z_i directed.

As an exercise, the reader may check that $(e_k^M)^{-1} = e_{2-k}^M$ where $2 - k$ is the charge on $L = L_1 \cup ... \cup L_m$ defined by $(2 - k)_i = 2 - k_i$ for $i = 1, ..., m$.

1.8. Proof of Lemma 1.3. In the case $\partial M = \emptyset$ Lemma 1.3 is essentially obvious. In this case $\partial\Sigma = \emptyset$ and the lemma amounts to saying that $c(e) (\mathrm{mod}\, 2)$ is a trivial homology class. Since M is parallelizable, there is a non-singular vector field u on M homotopic to $-u$. Clearly, $c([u]) = 1$ and the claim follows from the equality $c(e) \equiv c([u]) (\mathrm{mod}\, 2)$.

We now prove Lemma 1.3 in the case where $M = \mathrm{Z} = S^1 \times D^2$ is a solid torus. Let h be a generator of $H_1(\mathrm{Z})$. It is an elementary topological exercise to check that $b_0(\partial\Sigma) \equiv h \cdot \Sigma (\mathrm{mod}\, 2)$. Therefore it remains to prove that for any relative vector field u on Z the class $c([u])$ is an odd power of h. We define two residues $n_+(u), n_-(u) \in \mathbf{Z}/2\mathbf{Z}$ as follows. Choose a trivialization v_1 (resp. v_2, v_3) of the tangent bundle of S^1 (resp. of D^2) inducing the given orientation of S^1. This induces a trivialization (v_1, v_2, v_3) of the tangent bundle of Z. Restricting u to the disc $x \times D^2 \subset \mathrm{Z}$ with $x \in S^1$ and using the trivialization (v_1, v_2, v_3) we obtain a map f_u from D^2 to the unit 2-sphere $S^2 \subset \mathbf{R}^3$ such that $f_u(\partial D^2)$ does not meet the poles $(\pm 1, 0, 0)$ of S^2. Let $n_+(u)$ and $n_-(u)$ be the degrees mod 2 of f_u with respect to these poles. They depend only on $[u] \in \mathrm{vect}(\mathrm{Z})$. It is clear that $n_+(u) - n_-(u) = 1$. A simple calculation shows that $n_\pm(h^k u) = n_\pm(u) + k (\mathrm{mod}\, 2)$ for any $k \in \mathbf{Z}$. The restriction of v_1 to $\partial\mathrm{Z}$ is canonical and can therefore be used to describe a vector field representing $[u]^{-1}$. On the regular neighborhood of $\partial\mathrm{Z}$ used in the definition of $[u]^{-1}$, this vector field never belongs to $-\mathbf{R}_+ v_1$. Therefore $n_-(u^{-1}) = n_+(u) = n_-(u) + 1$. Hence $[u]^{-1} = h^k[u]$ with odd k and $c([u]) = [u]/[u]^{-1} = h^{-k}$ is an odd power of h.

Consider now the general case of Lemma 1.3. The intersection of Σ with a component T of ∂M is a system of disjoint simple closed curves homotopic to each other. We glue to M a solid torus Z_T along T so that the curves forming $\Sigma \cap T$ bound disjoint embedded discs in Z_T. Denote the union of these discs by Σ_T. If $\Sigma \cap T = \emptyset$ then we glue a solid torus Z_T to M along T in an arbitrary way and set $\Sigma_T = \emptyset$. Applying this procedure for all components T of ∂M we obtain a closed 3-manifold $\hat{M}$ and a closed embedded surface $\hat{\Sigma} = \Sigma \cup (\cup_T \Sigma_T) \subset \hat{M}$. Let $\hat{e}$ be the Euler structure on $\hat{M}$ obtained by gluing e with arbitrary Euler structures

$\{e_T \in \mathrm{vect}(\mathrm{Z}_T)\}_T$. We have $c(\hat{e}) = c(e) \cdot \prod_T c(e_T)$. Therefore

$$c(\hat{e}) \cdot \hat{\Sigma} = c(e) \cdot \Sigma + \sum_T c(e_T) \cdot \Sigma_T \,(\mathrm{mod}\, 2). \tag{1.8.a}$$

By the results of the previous paragraphs, $c(\hat{e}) \cdot \hat{\Sigma} = 0 (\mathrm{mod}\, 2)$ and

$$c(e_T) \cdot \Sigma_T = b_0(\partial \Sigma_T) = b_0(\partial \Sigma \cap T) \,(\mathrm{mod}\, 2).$$

Substituting this in (1.8.a), we obtain $c(e) \cdot \Sigma = b_0(\partial \Sigma) (\mathrm{mod}\, 2)$.

1.9. Remark. The Euler structures on a closed oriented 3-manifold M bijectively correspond to $Spin^c$-structures on the tangent vector bundle $TM \to M$ of M. The $Spin^c$-structure corresponding to the class $[u]$ of a non-singular vector field u on M is defined as follows. The 2-dimensional quotient vector bundle $TM/\mathbf{R}u$ is oriented and therefore has the structure group $U(1)$. This reduces the structure group of $TM = (TM/\mathbf{R}u) \oplus \mathbf{R}u$ to $U(1) = U(1) \oplus (1) \subset U(2) = Spin^c(3)$. Similarly, for a compact oriented 3-manifold M whose boundary consists of tori there is a bijection of $\mathrm{vect}(M)$ on the set of relative $Spin^c$-structures on M, see [Tu6].

2. Torsions of chain complexes

2.1. Basic definitions. Having two bases c, c' of a finite dimensional vector space, we can expand the vectors of c as linear combinations of the vectors of c'. The determinant of the resulting square matrix is denoted by $[c/c']$. The bases c, c' are *equivalent* if $[c/c'] = 1$.

Let $C = (C_m \to C_{m-1} \to \dots \to C_0)$ be a finite dimensional chain complex of length m over a field F. Suppose that for all $i = 0, 1, \dots, m$ both C_i and $H_i(C)$ are based, i.e., have a distinguished basis. In this situation one defines the torsion $\tau(C)$ as follows (cf. [Mi1], [Mi2]). Let c_i be the given basis in C_i and let h_i be a sequence of vectors in $\mathrm{Ker}(\partial_{i-1} : C_i \to C_{i-1})$ whose images under the projection $\mathrm{Ker}\, \partial_{i-1} \to H_i(C)$ form the given basis in $H_i(C)$. Let b_i be a sequence of vectors in C_i such that $\partial_{i-1}(b_i)$ is a basis in $\mathrm{Im}\, \partial_{i-1}$. Then for every i, the sequence $\partial_i(b_{i+1}) h_i b_i$ is a basis in C_i. Set

$$\tau(C) = \prod_{i=0}^{m} [\partial_i(b_{i+1}) h_i b_i / c_i]^{(-1)^{i+1}} \in F.$$

The torsion $\tau(C)$ depends on the equivalence classes of the given bases in $C_i, H_i(C)$ and does not depend on the choice of h_i, b_i. If C is acyclic then the definition of $\tau(C)$ simplifies since there is no need to fix a basis in $H_*(C)$.

We shall use a "sign-refined" torsion $\check{\tau}$ introduced in [Tu3]. Set

$$\alpha_i(C) = \dim C_i + \dim C_{i-1} + \dots + \dim C_0 \,(\mathrm{mod}\, 2) \in \mathbf{Z}/2\mathbf{Z},$$

$$\beta_i(C) = \dim H_i(C) + \dim H_{i-1}(C) + ... + \dim H_0(C) \, (\mathrm{mod}\, 2) \in \mathbf{Z}/2\mathbf{Z},$$

$$N(C) = \sum_{i=0}^{m} \alpha_i(C)\, \beta_i(C) \in \mathbf{Z}/2\mathbf{Z}.$$

Set $\check{\tau}(C) = (-1)^{N(C)}\tau(C) \in F$. If C is acyclic then $\check{\tau}(C) = \tau(C)$.

2.2. Relative torsions. Let $C = (C_m \to ... \to C_0)$ be a finite dimensional chain complex over a field F and let $C' = (C'_m \to ... \to C'_0)$ be a chain subcomplex of C. Suppose that the homology of C, C' and $C'' = C/C'$ are based. (The complexes themselves are not assumed to be based). We define a *relative torsion* $\tau(C' \subset C)$ as follows. Denote by $\mathcal{H}$ the homology sequence of the pair (C, C'):

$$\mathcal{H} = (H_m(C') \to H_m(C) \to H_m(C'') \to ... \to H_0(C) \to H_0(C'')).$$

Clearly, $\mathcal{H}$ is a based acyclic chain complex over F. Set

$$\tau(C' \subset C) = (-1)^{\theta(C,C')}\tau(\mathcal{H}) \in F$$

where

$$\theta(C, C') = \sum_{i=0}^{m} [(\beta_i(C) + 1)(\beta_i(C') + \beta_i(C'')) + \beta_{i-1}(C')\beta_i(C'')] \in \mathbf{Z}/2\mathbf{Z}.$$

The relative torsion $\tau(C' \subset C)$ can be expressed in terms of the torsions of C, C', C''. Namely, provide for all $i = 0, 1, ..., m$ the vector spaces $C_i, C'_i, C''_i = C_i/C'_i$ with compatible bases c_i, c'_i, c''_i where the compatibility means that c_i is equivalent to the basis $c'_i c''_i$ obtained as a juxtaposition of c'_i with a lift of c''_i to C_i. Then (see [Tu3], Lemma 3.4.2)

$$\tau(C' \subset C) = (-1)^{\nu(C,C')} \frac{\check{\tau}(C)}{\check{\tau}(C')\, \check{\tau}(C'')} \tag{2.2.a}$$

where

$$\nu(C, C') = \sum_{i=0}^{m} \alpha_i(C'')\, \alpha_{i-1}(C') \in \mathbf{Z}/2\mathbf{Z}. \tag{2.2.b}$$

If C, C' are acyclic, then $\tau(C' \subset C) = 1$ and (2.2.a) yields

$$\tau(C) = (-1)^{\nu(C,C')}\tau(C')\, \tau(C''). \tag{2.2.c}$$

2.3. Lemma. *Let $C^1 \subset C^2 \subset C^3$ be finite dimensional chain complexes of length m over a field F. Suppose that the homology of the chain complexes $C^1, C^2, C^3, C^2/C^1, C^3/C^1, C^3/C^2$ are based. Then*

$$\tau(C^1 \subset C^3) = \frac{\tau(C^1 \subset C^2)\, \tau(C^2 \subset C^3)}{\tau(C^2/C^1 \subset C^3/C^1)}. \tag{2.3.a}$$

Proof. Choose in each C_i^3 a basis whose first $\dim C_i^1$ vectors lie in C_i^1 and the next $\dim C_i^2 - \dim C_i^1$ vectors lie in C_i^2. This basis determines bases in $C_i^1, C_i^2, C_i^2/C_i^1, C_i^3/C_i^1, C_i^3/C_i^2$ in the obvious way. By (2.2.a), for any $1 \leq p < q \leq 3$,

$$\tau(C^p \subset C^q) = (-1)^{\nu(C^q, C^p)} \frac{\check{\tau}(C^q)}{\check{\tau}(C^p)\, \check{\tau}(C^q/C^p)},$$

$$\tau(C^2/C^1 \subset C^3/C^1) = (-1)^{\nu(C^3/C^1, C^2/C^1)} \frac{\check{\tau}(C^3/C^1)}{\check{\tau}(C^2/C^1)\, \check{\tau}(C^3/C^2)}.$$

Substituting these expressions in (2.3.a) we obtain that (2.3.a) is equivalent to

$$\nu(C^2, C^1) + \nu(C^3, C^1) + \nu(C^3, C^2) + \nu(C^3/C^1, C^2/C^1) = 0.$$

This formula follows from (2.2.b) and the equalities $\alpha_i(C^q/C^p) = \alpha_i(C^q) - \alpha_i(C^p)$ for all $1 \leq p < q \leq 3$. This implies (2.3.a).

3. Homology orientations of 3-manifolds

3.1. Homology orientations A *homology orientation* ω of a finite CW-pair (X, Y) is an orientation of the vector space $H_*(X, Y; \mathbf{R}) = \oplus_{i \geq 0} H_i(X, Y; \mathbf{R})$. We denote by $-\omega$ the opposite orientation.

The exterior, E, of an oriented link $L = L_1 \cup ... \cup L_m$ in an oriented 3-dimensional integral homology sphere N has a canonical homology orientation ω_L determined by the basis $([pt], t_1, ..., t_m, g_1, ..., g_{m-1})$ where $[pt]$ is the homology class of a point in E, $t_1, ..., t_m$ are the meridional generators of $H_1(E; \mathbf{R})$, and $g_1, ..., g_{m-1}$ are the generators of $H_2(E; \mathbf{R})$ represented by oriented boundaries of regular neighborhods of $L_1, ..., L_{m-1}$, respectively. (We use the "outward vector first" conveniton for the orientation of the boundary. The orientation in the regular neighborhoods is induced by the one in N). Note that ω_L does not depend on the numeration of the components of L and changes sign under inversion of the orientation of a component of L.

3.2. Induced homology orientations. Let M be a 3-manifold obtained by gluing m directed solid tori $\mathrm{Z}_1 = S^1 \times D_1, ..., \mathrm{Z}_m = S^1 \times D_m$ to a compact 3-manifold E. The aim of this section is to show that a homology orientation, ω, of E induces in a natural way a homology orientation, ω^M, of M.

Fix an orientation of the 2-discs $D_1, ..., D_m$ and provide each $\mathrm{Z}_i = S^1 \times D_i$ with orientation obtained as the product of the given orientation of its core and the fixed orientation in D_i. Clearly,

$$H_2(\mathrm{Z}_i, \partial \mathrm{Z}_i; \mathbf{R}) = \mathbf{R}\,[D_i, \partial D_i], \quad H_3(\mathrm{Z}_i, \partial \mathrm{Z}_i; \mathbf{R}) = \mathbf{R}\,[\mathrm{Z}_i, \partial \mathrm{Z}_i].$$

By excision, the vector space $H_*(M, E; \mathbf{R}) = \mathbf{R}^{2m}$ has a basis

$$[D_1, \partial D_1], ..., [D_m, \partial D_m], [\mathrm{Z}_1, \partial \mathrm{Z}_1], ..., [\mathrm{Z}_m, \partial \mathrm{Z}_m].$$

Denote the homology orientation of (M,E) determined by this basis by $\omega_{(M,E)}$. It is easy to check that $\omega_{(M,E)}$ depends neither on the choice of orientations in $D_1,...,D_m$ nor on the numeration of $\mathrm{Z}_1,...,\mathrm{Z}_m$.

There is a unique homology orientation $\tilde{\omega}$ of M such that the torsion of the exact homology sequence of the pair (M,E)

$$\mathcal{H} = (H_3(E;\mathbf{R}) \to H_3(M;\mathbf{R}) \to H_3(M,E;\mathbf{R}) \to ... \to H_0(E;\mathbf{R}) \to H_0(M;\mathbf{R}))$$

has a positive sign. It is understood that the torsion is taken with respect to arbitrary bases in $H_*(E;\mathbf{R}), H_*(M;\mathbf{R}), H_*(M,E;\mathbf{R})$ determining the orientations $\omega, \tilde{\omega}, \omega_{(M,E)}$, respectively. (The sign of $\tau(\mathcal{H}) \in \mathbf{R}\backslash\{0\}$ depends only on these orientations and does not depend on the choice of the bases). The homology orientation ω^M of M *induced by* ω is defined by

$$\omega^M = (-1)^{mb_3(M)+(b_0(E)+b_1(E))(b_0(M)+b_1(M)+m+1)+b_3(E)(b_3(M)+1)}\,\tilde{\omega}. \qquad (3.2.a)$$

The sign on the right-hand side is needed to ensure Lemma 3.3 below.

Clearly $(-\omega)^M = -(\omega^M)$. If $m=0$ then $M=E$ and $\omega^M = \tilde{\omega} = \omega$. In the case of connected E, formula (3.2.a) simplifies to

$$\omega^M = (-1)^{mb_3(M)+(b_1(E)+1)(b_1(M)+m)}\,\tilde{\omega} \qquad (3.2.b)$$

We establish two useful properties of induced homology orientations.

3.3. Lemma (transitivity). *Let E be a compact 3-manifold whose boundary consists of tori. Let $M \supset E$ be obtained by gluing directed solid tori to E. Let $M' \supset M$ be obtained by gluing directed solid tori to M. Then for any homology orientation ω of E,*

$$\omega^{M'} = (\omega^M)^{M'}.$$

Proof. Assume that M is obtained from E by gluing p directed solid tori and M' is obtained from M by gluing q directed solid tori. Fix a CW-decomposition of M' such that both M and E are its CW-subcomplexes. Consider the cellular chain complexes

$$c^1 = C_*(E;\mathbf{R}) \subset c^2 = C_*(M;\mathbf{R}) \subset c^3 = C_*(M';\mathbf{R}).$$

We fix orientations of the meridional discs of the solid tori forming $M'\backslash E$ and provide $H_*(c^2/c^1), H_*(c^3/c^1)$, $H_*(c^3/c^2)$ with bases as in Sect. 3.2. Let us provide $H_*(c^1), H_*(c^2)$ with bases determining the homology orientations ω, ω^M, respectively. The corresponding relative torsion $\tau(c^1 \subset c^2) \in \mathbf{R}\backslash\{0\}$ depends on the choice of the bases but its sign ± 1, denoted $\tau_0(c^1 \subset c^2, \omega, \omega^M)$, depends only on ω, ω^M. The definition of ω^M can be reformulated by saying that

$$\tau_0(c^1 \subset c^2, \omega, \omega^M)$$

$$= (-1)^{\theta(c^2,c^1)+pb_3(M)+(b_0(E)+b_1(E))(b_0(M)+b_1(M)+p+1)+b_3(E)(b_3(M)+1)}.$$

A direct computation shows that

$$\theta(c^2, c^1)$$

$$= pb_3(M)+(b_0(E)+b_1(E))(b_0(M)+b_1(M)+p+1)+b_3(E)(b_3(M)+1)+p \pmod 2.$$

This follows from the definition of $\theta(c^2, c^1)$ and the following equalities mod 2:

$$\beta_i(c^2/c^1) = \begin{cases} p, & if \ \ i = 2, \\ 0, & if \ \ i \neq 2, \end{cases}$$

$$\beta_i(c^1) = \begin{cases} b_0(E), \ if \ \ i = 0, \\ b_0(E) + b_1(E), \ if \ \ i = 1, \\ b_3(E), \ if \ \ i = 2, \\ 0, \ if \ \ i \geq 3, \end{cases} \qquad \beta_i(c^2) = \begin{cases} b_0(M) = b_0(E), \ if \ \ i = 0, \\ b_0(M) + b_1(M), \ if \ \ i = 1, \\ b_3(M), \ if \ \ i = 2, \\ 0, \ if \ \ i \geq 3. \end{cases}$$

(Here we use that $\chi(E) = \chi(\partial E)/2 = 0$ and $\chi(M) = \chi(\partial M)/2 = 0$.) Thus, the definition of ω^M can be reformulated by saying that

$$\tau_0(c^1 \subset c^2, \omega, \omega^M) = (-1)^p. \tag{3.3.a}$$

Similarly,

$$\tau_0(c^2 \subset c^3, \omega^M, (\omega^M)^{M'}) = (-1)^q, \ \ \tau_0(c^1 \subset c^3, \omega, \omega^{M'}) = (-1)^{p+q}.$$

It is easy to compute that $\theta(c^3/c^1, c^2/c^1) = 0$ and $\tau(c^2/c^1 \subset c^3/c^1) = +1$. Applying Lemma 2.3 and taking signs we obtain that

$$\tau_0(c^1 \subset c^3, \omega, (\omega^M)^{M'}) = \tau_0(c^1 \subset c^2, \omega, \omega^M)\, \tau_0(c^2 \subset c^3, \omega^M, (\omega^M)^{M'})$$

$$= (-1)^{p+q} = \tau_0(c^1 \subset c^3, \omega, \omega^{M'}).$$

Therefore $(\omega^M)^{M'} = \omega^{M'}$.

3.4. Lemma. *Let E be the exterior of an oriented link $L = L_1 \cup ... \cup L_m$ in an oriented 3-dimensional integral homology sphere N. Let $E' \supset E$ be the exterior of a non-void sublink L' of L. Then E' is obtained from E by gluing directed solid tori and $(\omega_L)^{E'} = \omega_{L'}$.*

Proof. Lemma 3.3 allows to deduce this lemma by induction from the case where L' has one component less than L. We however prefer to give a direct proof in the general case. Assume for concreteness that $L' = L_1 \cup ... \cup L_n$ with $1 \leq n \leq m$. It is clear that E' is obtained from E by gluing $m - n$ directed solid tori $\mathrm{Z}_{n+1}, ..., \mathrm{Z}_m$ which are regular neighborhoods of $L_{n+1}, ..., L_m$, respectively. We orient the meridional disc of each Z_i so that its intersection index with L_i equals

+1. This yields a basis in $H_*(E', E)$ as in Sect. 3.2. The homology $H_*(E), H_(E')$ have bases described in Sect. 3.1. The exact homology sequence, $\mathcal{H}$, of the pair (E', E) splits as a concatenation of two short exact sequences

$$H_3(E', E) \to H_2(E) \to H_2(E'),\ H_2(E', E) \to H_1(E) \to H_1(E')$$

and the inclusion isomorphism $H_0(E) \to H_0(E')$. The torsions of these pieces with respect to the choosen bases are equal to $(-1)^{mn+n}, (-1)^{mn+n}$ and $+1$, respectively. Hence $\tau(\mathcal{H}) = +1$ and therefore $\tilde{\omega}_L = \omega_{L'}$. The numerical expression appearing in (3.2.b) equals

$$(m-n)b_3(E') + (b_1(E)+1)(b_1(E') + m - n) \equiv 0 \,(\mathrm{mod}\, 2).$$

Therefore the sign in (3.2.b) is + and $(\omega_L)^{E'} = \tilde{\omega}_L = \omega_{L'}$.

4. Combinatorial Euler structures on 3-manifolds

4.1. Combinatorial Euler structures. In this and the next two subsections we recall the theory of combinatorial Euler structures and their torsions following [Tu4]. Let X be a finite CW space. An *Euler chain* in X is a 1-dimensional singular chain ξ in X with

$$\partial \xi = \sum_a (-1)^{\dim a} \alpha_a$$

where a runs over all (open) cells of X and α_a is a point in a. For two Euler chains ξ, η we define a "quotient" $\xi/\eta \in H_1(X)$ as follows. Let

$$\partial \xi = \sum_a (-1)^{\dim a} \alpha_a \quad \text{and} \quad \partial \eta = \sum_a (-1)^{\dim a} \beta_a$$

where $\alpha_a, \beta_a \in a$. For each cell a, choose a path $\gamma_a : [0,1] \to a$ from α_a to β_a. Then $\xi/\eta \in H_1(X)$ is the homology class of the singular 1-cycle $\xi - \eta + \sum_a (-1)^{\dim a} \gamma_a$.

We say that two Euler chains ξ, η in X define the same *(combinatorial) Euler structure* on X if $\xi/\eta = 1$. The Euler structure represented by ξ is denoted by $[\xi]$. The set of Euler structures on X is denoted by $\mathrm{Eul}(X)$. This set is non-void iff for every connected component X_0 of X we have $\chi(X_0) = 0$. It is clear that $\mathrm{Eul}(X) = \prod_{X_0} \mathrm{Eul}(X_0)$ where X_0 runs over the components of X.

The group $H_1(X)$ acts on $\mathrm{Eul}(X)$: if $[h] \in H_1(X)$ is the homology class of a 1-cycle h and ξ is an Euler chain on X then $[h][\xi] = [\xi + h] \in \mathrm{Eul}(X)$. If $\mathrm{Eul}(X) \neq \emptyset$ then this action is free and transitive.

For any cellular subdivision X' of X there is a canonical $H_1(X)$-equivariant bijection $\mathrm{Eul}(X) \to \mathrm{Eul}(X')$. This allows us to define the set of combinatorial Euler structures $\mathrm{Eul}(M)$ for a smooth compact manifold M. This set is obtained

by identification of the sets $\{\mathrm{Eul}(X)\}_X$ where X runs over the C^1-triangulations of M. By [Tu4], there is a canonical $H_1(M)$-equivariant bijection $\mathrm{Eul}(M)=\mathrm{vect}(M)$.

4.2. Torsions of Euler structures. Let X be a finite connected CW space with $\chi(X)=0$. Let F be a field and $\varphi:\mathbf{Z}[H_1(X)]\to F$ be a ring homomorphism. For every Euler structure $e\in\mathrm{Eul}(X)$ and every homology orientation ω of X we define a torsion $\tau^\varphi(X,e,\omega)\in F$.

Consider the maximal abelian covering $\tilde X$ of X with its induced CW structure. The group $H=H_1(X)$ acts on $\tilde X$ via the covering transformations permuting the cells in $\tilde X$ lying over any given cell in X. A family of cells in $\tilde X$ is said to be *fundamental* if over each cell of X lies exactly one cell of this family. Every fundamental family of cells in $\tilde X$ gives rise to an Euler structure on X: consider a spider in $\tilde X$ formed by arcs in $\tilde X$ connecting a point $x\in\tilde X$ to points in these cells; the arc joining x to a point of an odd-dimensional (resp. even-dimensional) cell should be oriented towards x (resp. out of x). Projecting this spider to X we obtain an Euler chain in X representing an element of $\mathrm{Eul}(X)$. It is clear that any Euler structure on X arises in this way from a fundamental family of cells in $\tilde X$. We fix a fundamental family of cells $\tilde e$ in $\tilde X$ corresponding to $e\in\mathrm{Eul}(X)$ in this way.

We orient and order the cells in the family $\tilde e$ in an arbitrary way. This yields a basis for the cellular chain complex $C_*(\tilde X)=C_*(\tilde X;\mathbf{Z})$ over $\mathbf{Z}[H]$. Consider the induced basis in the chain complex over F

$$C_*^\varphi(X)=F\otimes_{\mathbf{Z}[H]}C_*(\tilde X)$$

where $\mathbf{Z}[H]$ acts on F via φ. If this based chain complex is acyclic, i.e., $H_*^\varphi(X)=H_*(C_*^\varphi(X))=0$, then we can consider its torsion $\tau(C_*^\varphi(X))\in F\backslash 0$. Since the cells of $\tilde e$ are in bijective correspondence with the cells of X, the choosen orientation and order for the cells of $\tilde e$ induce an orientation and an order for the cells of X. This yields a basis of the cellular chain complex $C_*(X;\mathbf{R})$ over $\mathbf{R}$. Provide the homology of $C_*(X;\mathbf{R})$ with a basis determining the homology orientation ω. Consider the sign-refined torsion $\check\tau(C_*(X;\mathbf{R}))\in\mathbf{R}\backslash\{0\}$ of the resulting based chain complex with based homology. We need only its sign ±1 denoted $\check\tau_0(C_*(X;\mathbf{R}))$. Set

$$\tau^\varphi(X,e,\omega)=\begin{cases}\check\tau_0(C_*(X;\mathbf{R}))\,\tau(C_*^\varphi(X))\in F\backslash\{0\},\ if\ \ H_*^\varphi(X)=0,\\ 0\in F,\ if\ \ H_*^\varphi(X)\neq 0.\end{cases}$$

It turns out that $\tau^\varphi(X,e,\omega)\in F$ does not depend on the auxiliary choices and is invariant under cellular subdivisions of X. We have

$$\tau^\varphi(X,e,-\omega)=-\tau^\varphi(X,e,\omega)\ \text{ and }\ \tau^\varphi(X,he,\omega)=\varphi(h)\,\tau^\varphi(X,e,\omega)\qquad(4.2.a)$$

for any $e\in\mathrm{Eul}(X),h\in H_1(X)$,

We shall use the notation $\tau^\varphi(X,e)$ for $\pm\tau^\varphi(X,e,\omega)$:

$$\tau^\varphi(X,e)=\pm\tau^\varphi(X,e,\omega)=\tau^\varphi(X,e,\pm\omega).$$

We will view $\tau^\varphi(X,e)$ as an element of F defined up to multiplication by -1.

The definition of $\tau^\varphi(X,e,\omega)$ extends by multiplicativity to the case of a non-connected finite CW space X. All the connected components of X should be homology oriented. Then the torsion corresponding to $e \in \mathrm{Eul}(X)$ and a ring homomorphism $\varphi : \mathbf{Z}[H_1(X)] \to F$ is defined as the product of the torsions of the components of X corresponding to the restrictions of e and φ.

In Section 6 we shall need a version $\check{\tau}^\varphi$ of τ^φ which is always non-zero. Assume that the vector space $H_*^\varphi(X)$ is endowed with a basis b and set

$$\check{\tau}^\varphi(X,e,\omega;b) = \check{\tau}_0(C_*(X;\mathbf{R}))\,\check{\tau}(C_*^\varphi(X),b) \in F$$

where $\check{\tau}_0(C_*(X;\mathbf{R})) = \pm 1$ is the same sign as above and $\check{\tau}(C_*^\varphi(X),b)$ is the torsion $\check{\tau}$ of $C_*^\varphi(X)$ corresponding to the basis b in homology. The bases in the vector spaces of chains are determined by $\tilde{e}$ as above. The torsion $\check{\tau}^\varphi(X,e,\omega;b)$ does not depend on the auxiliary choices, never equals 0 and is invariant under cellular subdivisions of X. If $H_*^\varphi(X)=0$ then $\check{\tau}^\varphi(X,e,\omega;\emptyset) = \tau^\varphi(X,e,\omega)$.

4.3. Example. The 2-torus $T = S^1 \times S^1$ has a canonical Euler structure, e_T^{can}, defined as follows. Set $H = H_1(T) = \mathbf{Z}^2$. Consider the inclusion μ of the group ring $\mathbf{Z}[H]$ into its field of quotients $Q(H)$. It is easy to check (using for instance a CW-decomposition of T consisting of one 0-cell, two 1-cells and one 2-cell) that the chain complex $C_*^\mu(T)$ is acyclic and for any $e \in \mathrm{Eul}(T)$, $\tau^\mu(T,e) = \pm g_e$ with $g_e \in H$. By (4.2.a), $g_{he} = hg_e$ for all $h \in H$. Therefore the formula $e \mapsto g_e$ establishes a bijection $\mathrm{Eul}(T) \to H$ natural with respect to diffeomorphisms $T \to T$. The Euler structure $e \in \mathrm{Eul}(T)$ with $g_e = 1 \in H$ is denoted e_T^{can}. This is the unique Euler structure on T invariant under all diffeomorphisms $T \to T$. Therefore under the identification $\mathrm{Eul}(T) = \mathrm{vect}(T)$, e_T^{can} corresponds to the class $[w]$ described in Sect. 1.1.

One can check using the CW-decomposition of T as above, that for any field F and any ring homomorphism $\varphi : \mathbf{Z}[H_1(T)] \to F$ such that $\varphi(H_1(T)) \neq 1$, we have

$$\tau^\varphi(T, e_T^{\mathrm{can}}) = \pm 1. \tag{4.3.a}$$

4.4. Gluing of combinatorial Euler structures. Let M be a compact 3-manifold whose boundary consists of tori. Let $T \subset M \backslash \partial M$ be a finite system of disjoint embedded 2-tori splitting M into two 3-manifolds M_0, M_1. We define a gluing map $\cup : \mathrm{Eul}(M_0) \times \mathrm{Eul}(M_1) \to \mathrm{Eul}(M)$ as follows. Fix a CW-decomposition of M so that M_0, M_1, T are its CW-subcomplexes. If ξ_0, ξ_1 are Euler chains on M_0, M_1, respectively, then we set

$$[\xi_0] \cup [\xi_1] = [\xi_0 + \xi_1 - \xi] \in \mathrm{Eul}(M)$$

where ξ is an Euler chain on T representing the canonical Euler structure on each component of T, cf. Sect. 4.3. Note that $\xi_0 + \xi_1 - \xi$ is an Euler chain on M.

It is clear that the gluing operation $\cup$ satisfies (1.5.b) where $e_0 \in \mathrm{Eul}(M_0), e_1 \in \mathrm{Eul}(M_1)$. It can be deduced from definitions that the diagram

$$\begin{array}{ccc}
\mathrm{Eul}(M_0) \times \mathrm{Eul}(M_1) & \xrightarrow{\cup} & \mathrm{Eul}(M) \\
{\scriptstyle =}\downarrow & & \downarrow{\scriptstyle =} \\
\mathrm{vect}(M_0) \times \mathrm{vect}(M_1) & \xrightarrow{\cup} & \mathrm{vect}(M)
\end{array}$$

is commutative.

4.5. Lemma. *Let M, T, M_0, M_1 be the same objects as in Sect. 4.4. Let F be a field and $\varphi : \mathbf{Z}[H_1(M)] \to F$ be a ring homomorphism such that for every component T' of T, we have $\varphi(H_1(T')) \neq 1$. Let $\mathrm{in}_i : \mathbf{Z}[H_1(M_i)] \to \mathbf{Z}[H_1(M)]$ be the ring homomorphism induced by the inclusion homomorphism $H_1(M_i) \to H_1(M)$ where $i = 0, 1$. Then for any $e_0 \in \mathrm{Eul}(M_0), e_1 \in \mathrm{Eul}(M_1)$,*

$$\tau^{\varphi}(M, e_0 \cup e_1) = \tau^{\varphi\, \mathrm{in}_0}(M_0, e_0)\ \tau^{\varphi\, \mathrm{in}_1}(M_1, e_1) \in F/\{\pm 1\}. \qquad (4.5.a)$$

Proof. Let us assume for simplicity that M_0, M_1 are connected and T is a 2-torus; the general case is similar. Fix a CW-decomposition of M so that M_0, M_1, T are its CW-subcomplexes. Denote the inclusion homomorphism $\mathbf{Z}[H_1(T)] \to \mathbf{Z}[H_1(M)]$ by in. We have the usual short exact sequence of chain complexes

$$0 \to C_*^{\varphi\, \mathrm{in}}(T) \to C^{\varphi\, \mathrm{in}_0}(M_0) \oplus C^{\varphi\, \mathrm{in}_1}(M_1) \to C_*^{\varphi}(M) \to 0. \qquad (4.5.b)$$

The assumption $\varphi(H_1(T)) \neq 1$ implies that $C_*^{\varphi\, \mathrm{in}}(T)$ is acyclic. If $C_*^{\varphi}(M)$ is not acyclic then at least one of the complexes $C^{\varphi\, \mathrm{in}_0}(M_0), C^{\varphi\, \mathrm{in}_1}(M_1)$ is not acyclic and both sides of (4.5.a) are equal to 0. Assume that $C_*^{\varphi}(M)$ is acyclic. Then all the complexes involved are acyclic and the torsions in (4.5.a) are non-zero.

We now describe the gluing of combinatorial Euler structures in terms of fundamental families of cells. Let $\tilde{T}, \tilde{M}_0, \tilde{M}_1, \tilde{M}$ be the maximal abelian coverings of T, M_0, M_1, M, respectively. We have a commutative diagram of cellular maps

$$\begin{array}{ccc}
\tilde{T} & \xrightarrow{f_0} & \tilde{M}_0 \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle g_0} \\
\tilde{M}_1 & \xrightarrow{g_1} & \tilde{M}.
\end{array}$$

Each of these maps is a lift of the corresponding inclusion and is equivariant with respect to the inclusion homomorphism in 1-homology. For instance, the map $\tilde{T} \to \tilde{M}_0$ is a lift of the inclusion $T \hookrightarrow M_0$ and is equivariant with respect to the inclusion homomorphism $H_1(T) \to H_1(M_0)$. Choose a fundamental family of cells $\tilde{e}$ in $\tilde{T}$ representing e_T^{can}. It is clear that $f_0(\tilde{e})$ is a family of cells in $\tilde{M}_0$ such that over each cell of $T \subset M_0$ lies exactly one cell of this family. We can add to $f_0(\tilde{e})$

a certain set $\tilde{e}_0$ of cells in $\tilde{M}_0 \backslash f_0(\tilde{T})$ so that $f_0(\tilde{e}) \cup \tilde{e}_0$ is a fundamental family of cells in $\tilde{M}_0$ representing e_0. Choose also a fundamental family of cells $\tilde{e}_1$ in $\tilde{M}_1$ representing e_1. Then $g_0(\tilde{e}_0) \cup g_1(\tilde{e}_1)$ is a fundamental family of cells in $\tilde{M}$. It follows from definitions that this family represents $e_0 \cup e_1$.

The families of cells $\tilde{e}$, $f_0(\tilde{e}) \cup \tilde{e}_0$, $\tilde{e}_1$, and $g_0(\tilde{e}_0) \cup g_1(\tilde{e}_1)$ determine bases in the chain complexes $C_*^{\varphi\, \mathrm{in}}(T), C^{\varphi\, \mathrm{in}_i}(M_i)$ with $i = 0, 1$, and $C_*^{\varphi}(M)$, respectively. It follows from (4.3.a) and definitions that the corresponding torsions are equal to ± 1, $\tau^{\varphi\, \mathrm{in}_i}(M_i, e_i)$ with $i = 0, 1$ and $\tau^{\varphi}(M, e_0 \cup e_1)$. These bases in $C_*^{\varphi\, \mathrm{in}}(T), C^{\varphi\, \mathrm{in}_0}(M_0) \oplus C^{\varphi\, \mathrm{in}_1}(M_1)$, and $C_*^{\varphi}(M)$ are compatible in the sense of Sect. 2.2 (at least up to sign). Now, formula (2.2.c) implies (4.5.a).

4.6. Duality for torsions. One of the fundamental properties of the Reidemeister torsions is the duality due to Franz [Fr] and Milnor [Mi1]. We state the duality theorem for torsions of Euler structures on 3-manifolds following [Tu4]. Let M be a compact orientable 3-manifold whose boundary consists of tori. Let F be a field with involution $f \mapsto \overline{f} : F \to F$ and $\varphi : \mathbf{Z}[H_1(M)] \to F$ be a ring homomorphism such that $\overline{\varphi(h)} = \varphi(h^{-1})$ for any $h \in H_1(M)$. Then for every $e \in \mathrm{vect}(M) = \mathrm{Eul}(M)$,

$$\overline{\tau^{\varphi}(M, e)} = \tau^{\varphi}(M, e^{-1}) = \varphi(c(e))^{-1}\, \tau^{\varphi}(M, e). \qquad (4.6.a)$$

All torsions here are elements of F defined up to multiplication by -1. For closed 3-manifolds formula (4.6.a) is established in [Tu4], Appendix B. Here we use this result to prove (4.6.a) in the case $\partial M \neq \emptyset$. Denote by K the double of M obtained by gluing two copies of M along the identity homeomorphism of the boundaries. Clearly, K is a closed orientable 3-manifold. Let $\psi : \mathbf{Z}[H_1(K)] \to \mathbf{Z}[H_1(M)]$ be the ring homomorphism induced by the natural folding map $K \to M$ which is the identity on both copies of M. The restrictions of $\varphi\psi$ to both copies of M are equal to φ. Applying (4.6.a) to the Euler structure $e \cup e$ on K we obtain

$$\overline{\tau^{\varphi\psi}(K, e \cup e)} = \tau^{\varphi\psi}(K, (e \cup e)^{-1}) = \tau^{\varphi\psi}(K, e^{-1} \cup e^{-1}).$$

By Lemma 4.5,

$$\tau^{\varphi\psi}(K, e \cup e) = (\tau^{\varphi}(M, e))^2, \quad \tau^{\varphi\psi}(K, e^{-1} \cup e^{-1}) = (\tau^{\varphi}(M, e^{-1}))^2.$$

Thus,

$$\overline{\tau^{\varphi}(M, e)}^{\,2} = (\tau^{\varphi}(M, e^{-1}))^2$$

and therefore $\overline{\tau^{\varphi}(M, e)} = \tau^{\varphi}(M, e^{-1})$.

In Appendix 3 we discuss a more precise version of duality taking into account homology orientations.

5. The Torres formula for torsions

5.1. Lemma. *Let E be a compact connected 3-manifold such that ∂E consists of tori. Let M be a 3-manifold obtained by gluing a directed solid torus* Z *to E and let $h \in H_1(M)$ be the homology class of the core of* Z. *Let F be a field and $\varphi : \mathbf{Z}[H_1(E)] \to F$ be a ring homomorphism inducing a ring homomorphism $\varphi^M : \mathbf{Z}[H_1(M)] \to F$. Then for any $e \in \mathrm{Eul}(E)$ and any homology orientation ω of E,*

$$\tau^{\varphi}(E, e, \omega) = (\varphi^M(h) - 1)\, \tau^{\varphi^M}(M, e^M, \omega^M). \tag{5.1.a}$$

The condition on φ means that φ maps the homology class of the boundary of the meridional disc of Z to 1. Lemma 5.1 is a version of the Torres identity for the Alexander polynomials of links in S^3.

We prove Lemma 5.1 in Sect. 5.3 after a few preliminary computations.

5.2. Solid torus re-examined. Let $\mathrm{Z} = S^1 \times D^2$ be a directed solid torus endowed with a CW-decomposition obtained from a certain CW-decomposition of $\partial \mathrm{Z}$ by adjoining two cells: a 2-cell $a^2 = x \times D^2$ with $x \in S^1$ and a 3-cell a^3 with interior $(S^1 \backslash \{x\}) \times \mathrm{Int}(D^2)$. Let $\tilde{\mathrm{Z}} = \mathbf{R} \times D^2$ be the universal covering of Z with induced CW-decomposition. We shall exhibit a fundamental family of cells in $\tilde{\mathrm{Z}}$ representing the distinguished Euler structure e_{Z} on Z, cf. Sect. 1.4.

Choose a fundamental family of cells in the maximal abelian covering of $\partial \mathrm{Z}$ representing $e^{\mathrm{can}}_{\partial \mathrm{Z}} \in \mathrm{Eul}(\partial \mathrm{Z})$, cf. Sect. 4.3. Projecting this family to $\partial \tilde{\mathrm{Z}}$ we obtain a family of cells, $\tilde{e}$, in $\partial \tilde{\mathrm{Z}}$ such that over each cell of $\partial \mathrm{Z}$ lies exactly one cell of $\tilde{e}$. We lift a^2, a^3 to cells $\tilde{a}^2, \tilde{a}^3$ of $\tilde{\mathrm{Z}}$ such that $\partial \tilde{a}^3 = \pm(h-1)\tilde{a}^2$ modulo 2-cells lying in $\partial \tilde{\mathrm{Z}}$. Here $h = h_{\mathrm{Z}} \in H_1(\mathrm{Z})$ is the distinguished generator. The family of cells $\tilde{e}, \tilde{a}^2, \tilde{a}^3$ in $\tilde{\mathrm{Z}}$ is fundamental and represents a certain Euler structure $e \in \mathrm{Eul}(\mathrm{Z})$. We claim that $e = e_{\mathrm{Z}}$.

We need to check that $c(e) = h^{-1}$. One possible proof consists in a combinatorial computation of $c(e)$ from a fundamental family of cells (or from an Euler chain) representing e. We shall use another approach based on (4.6.a). Set $H = H_1(\mathrm{Z})$ and consider the inclusion μ of the group ring $\mathbf{Z}[H]$ into its field of quotients $Q(H)$. We claim that

$$\tau^{\mu}(\mathrm{Z}, e) = \pm 1/(h-1). \tag{5.2.a}$$

This would imply the equality $c(e) = h^{-1}$ since by (4.6.a),

$$\pm c(e) = \pm \mu(c(e)) = \tau^{\mu}(M, e) \,/\, \overline{\tau^{\mu}(M, e)} = \pm(h^{-1} - 1)/(h-1) = \pm h^{-1}$$

where the overline denotes the involution in $Q(H)$ sending each $g \in H$ to g^{-1}.

To prove (5.2.a) we consider a short exact sequence of acyclic chain complexes

$$0 \to C^{\mu\,\mathrm{in}}_*(\partial \mathrm{Z}) \to C^{\mu}_*(\mathrm{Z}) \to C^{\mu}_*(\mathrm{Z}, \partial \mathrm{Z}) \to 0$$

where $\mathrm{in} : \mathbf{Z}[H_1(\partial \mathrm{Z})] \to \mathbf{Z}[H]$ is the inclusion homomorphism. We orient and order the cells of $\tilde{e}$ so that they determine a basis of $C^{\mu\,\mathrm{in}}_*(\partial \mathrm{Z})$. By (4.3.a), we have $\tau(C^{\mu\,\mathrm{in}}_*(\partial \mathrm{Z})) = \tau^{\mu\,\mathrm{in}}(\partial \mathrm{Z}, e^{\mathrm{can}}_{\partial \mathrm{Z}}) = \pm 1$. The cells $\tilde{a}^2, \tilde{a}^3$ endowed with arbitrary

orientations yield a basis for $C_*^\mu(\mathrm{Z}, \partial \mathrm{Z}) = (Q(H)\,\tilde a^3 \to Q(H)\,\tilde a^2)$. The torsion of this chain complex is equal to $\pm(h-1)^{-1}$. The family $\tilde e, \tilde a^2, \tilde a^3$ determines a basis in $C_*^\mu(\mathrm{Z})$. By (2.2.c),

$$\tau^\mu(\mathrm{Z}, e) = \pm\tau(C_*^\mu(\mathrm{Z})) = \pm\tau(C_*^{\mu\,\mathrm{in}}(\partial \mathrm{Z}))\,\tau(C_*^\mu(\mathrm{Z}, \partial\mathrm{Z})) = \pm(h-1)^{-1}.$$

5.3. Proof of Lemma 5.1. We fix a CW-decomposition of E (such that ∂E is a subcomplex) and extend it to a CW-decomposition of M by adjoining two cells $a^2, a^3 \subset \mathrm{Z}$ as in Sect. 5.2. Choose a fundamental family of cells in the maximal abelian covering of E representing $e \in \mathrm{Eul}(E)$. Projecting these cells to the maximal abelian covering, $\tilde M$, of M we obtain a family of cells, $\tilde e$, in $\tilde M$ such that over each cell of E lies exactly one cell of $\tilde e$. We lift a^2, a^3 to cells $\tilde a^2, \tilde a^3 \subset \tilde M$ so that $\partial \tilde a^3 = \pm(h-1)\,\tilde a^2$ modulo 2-cells lying over $\partial \mathrm{Z}$. It is clear that $\tilde e, \tilde a^2, \tilde a^3$ is a fundamental family of cells in $\tilde M$. As in Sect. 4.5, one can deduce from the results of Sect. 5.2 that this family represents $e^M = e \cup e_{\mathrm{Z}}$. We orient and order the cells belonging to $\tilde e$ in an arbitrary way. The cell a^2 is a meridional disc of Z, we orient it in an arbitrary way and lift this orientation to $\tilde a^2$. The product orientation of $\mathrm{Z} = S^1 \times a^2$ induces orientations in a^3 and $\tilde a^3$. With these orientations, $\partial \tilde a^3 = (h-1)\,\tilde a^2$ modulo 2-cells lying over $\partial \mathrm{Z}$.

Consider the chain complexes C, $C' \subset C$ and C'' defined by

$$C' = C_*^\varphi(E),\ C = C_*^{\varphi^M}(M),\ C'' = C/C'. \tag{5.3.a}$$

The family of cells $\tilde e, \tilde a^2, \tilde a^3$ determines bases in C, C', C'' in a compatible way. The non-zero part of C'' amounts to the boundary homomorphism $F\tilde a^3 \to F\tilde a^2$ sending $\tilde a^3$ to $(\varphi^M(h)-1)\,\tilde a^2$. Now we consider two cases.

Case $\varphi^M(h) \neq 1$. In this case C'' is acyclic and $\tau(C'') = (\varphi^M(h)-1)^{-1}$. The complexes C and C' are acyclic or not simultaneously. If they are not acyclic then both sides of (5.1.a) are equal to 0. Assume that they are acyclic. By (2.2.c),

$$\tau(C) = (-1)^{\nu(C,C')}\tau(C')\,\tau(C'') = (-1)^{\nu(C,C')}\tau(C')\,(\varphi^M(h)-1)^{-1}. \tag{5.3.b}$$

Consider the chain complexes c, $c' \subset c$ and c'' defined by

$$c' = C_*(E; \mathbf{R}),\ c = C_*(M; \mathbf{R}),\ c'' = c/c' = C_*(M, E; \mathbf{R}). \tag{5.3.c}$$

The same ordered families of oriented cells as in the previous paragraph determine bases in c', c, c''. We provide their homology with bases determining the homology orientations $\omega, \omega^M, \omega_{(M,E)}$, respectively (cf. Sect. 3.2). By (2.2.a),

$$\check\tau_0(c) = (-1)^{\nu(c,c')}\,\check\tau_0(c')\,\check\tau_0(c'')\,\tau_0(c' \subset c)$$

where the subindex 0 indicates that we consider the signs of the corresponding torsions. Observe that $\nu(c,c') = \nu(C,C')$ and by (3.3.a), $\tau_0(c' \subset c) = -1$. The

non-trivial part of c'' amounts to a zero boundary homomorphism $\mathbf{R}\, a^3 \to \mathbf{R}\, a^2$. The oriented cells a^2, a^3 determine a basis in $H_*(c''_2) = H_*(M, E; \mathbf{R})$ representing $\omega_{(M,E)}$. Hence $\tau_0(c'') = 1$, $N(c'') = 1$, and

$$\check{\tau}_0(c'') = (-1)^{N(c'')} \tau_0(c'') = -1.$$

We conclude that

$$\check{\tau}_0(c) = (-1)^{\nu(C,C')}\, \check{\tau}_0(c').$$

Multiplying this by (5.3.b), we obtain

$$\tau^\varphi(E, e, \omega) = \check{\tau}_0(c')\, \tau(C') = (\varphi^M(h) - 1)\, \check{\tau}_0(c)\, \tau(C)$$

$$= (\varphi^M(h) - 1)\, \tau^{\varphi^M}(M, e^M, \omega^M).$$

Case $\varphi^M(h) = 1$. We need to show that $\tau^\varphi(E, e, \omega) = 0$ or equivalently that $H_*^\varphi(E) \neq 0$. Let us assume that $H_*(C') = H_*^\varphi(E) = 0$ and look for a contradiction. It is clear that $H_i(C'') = F$ for $i = 2, 3$ and $H_i(C'') = 0$ for $i \neq 2, 3$. The exact homology sequence of the pair (C, C') implies that $H_i(C) = F$ for $i = 2, 3$ and $H_i(C) = 0$ for $i \neq 2, 3$. If $\partial M \neq \emptyset$ then M collapses onto a 2-dimensional subcomplex which contradicts $H_3(C) = F$. If M is closed then it has a CW-decomposition with only one 3-cell. For an appropriate choice of bases in C_2, C_3, the matrix of the boundary homomorphism $C_3 \to C_2$ is then a row $[\varphi^M(g_1) - 1, ..., \varphi^M(g_m) - 1]$ where $g_1, ..., g_m$ are the generators of $H_1(M)$ dual to the 2-cells. The equality $H_3(C) = F$ implies that $\varphi^M(g_1) = ... = \varphi^M(g_m) = 1$ so that $\varphi^M(H_1(M)) = 1$. However, in this case $H_0(C) = F$ which is a contradiction.

6. Additivity of torsions

6.1. The setting. Let E be a compact connected oriented 3-manifold such that ∂E consists of tori. Let T be a component of ∂E and α be an oriented non-contractible simple closed curve on T. Fix $\varepsilon = \pm 1$. We can glue a solid torus Z to E along T so that α bounds a meridional disc in Z. The resulting 3-manifold, M, depends only on α. The orientation of E extends to M. We make $\mathrm{Z} = \overline{M \backslash E}$ directed by first orienting its meridional disc, a^2, so that $\partial a^2 = \alpha$ in the oriented category and then orienting the core circle of Z so that the orientation in M restricted to Z is ε times the product orientation. Thus, the manifold M is obtained from E by gluing a directed solid torus. We shall denote this manifold by $M_{E,\alpha,\varepsilon}$. Note that the manifolds $M_{E,\alpha,\pm\varepsilon}$ have the same underlying manifold M and differ only by the structure of a directed solid torus on $\overline{M \backslash E}$.

Fix a ring homomorphism φ from $\mathbf{Z}[H_1(E)]$ to a field F such that $\varphi(H_1(T)) = 1$. There is a (unique) ring homomorphism $\varphi^M : \mathbf{Z}[H_1(M)] \to F$ whose composition with the inclusion homomorphism $\mathbf{Z}[H_1(E)] \to \mathbf{Z}[H_1(M)]$ is equal to φ.

6.2. Lemma. *Let $E, T, \alpha, \varepsilon, M = M_{E,\alpha,\varepsilon}, F, \varphi, \varphi^M$ be as in Sect. 6.1. Suppose that $H_*^{\varphi^M}(M) = 0$. Let $\hat{E} \to E$ be the regular covering of E corresponding to*

the group $H_1(E) \cap \varphi^{-1}(1) \subset H_1(E)$. Then T lifts to a torus $\hat{T} \subset \partial \hat{E}$ and α lifts to an oriented simple closed curve $\hat{\alpha}$ on $\hat{T}$ such that their homology classes $[\hat{\alpha}] \in H_1^{\varphi}(E), [\hat{T}] \in H_2^{\varphi}(E)$ form a basis in $H_*^{\varphi}(E)$. (Here the orientation of $T, \hat{T}$ is induced by the one in E). For any homology orientation ω of E and any $e \in \mathrm{Eul}(E)$,

$$\tau^{\varphi^M}(M, e^M, \omega^M) = -\varepsilon\, \tau^{\varphi}(E, e, \omega; [\hat{\alpha}], [\hat{T}]). \tag{6.2.a}$$

Proof. We fix a CW-decomposition of E (such that α and T are subcomplexes) and extend it to a CW-decomposition of M by adjoining two cells $a^2, a^3 \subset \mathrm{Z}$ as in Sect. 5.2. We can assume that $\partial a^2 = \alpha$. We orient a^2 so that $\partial a^2 = \alpha$ in the oriented category. We provide a^3 with the orientation determined by the product orientation in Z. (This orientation in a^3 depends on ε). Let $\tilde{e}, \tilde{a}^2, \tilde{a}^3$ be a fundamental family of ordered oriented cells in the maximal abelian covering $\tilde{M}$ of M constructed as in Sect. 5.3 and representing $e^M = e \cup e_{\mathrm{Z}}$.

Consider the chain complexes C, C', C'' defined by (5.3.a). We shall apply to them formula (2.2.a). The family of cells $\tilde{e}, \tilde{a}^2, \tilde{a}^3$ defines compatible bases in C, C', C'' in the usual way. The homology of C, C', C'' are provided with bases as follows. By assumption, $H_*(C) = H_*^{\varphi^M}(M) = 0$. Since $\varphi^M(H_1(\mathrm{Z})) = 1$, the non-trivial part of C'' amounts to a zero homomorphism $F\tilde{a}^3 \to F\tilde{a}^2$. Hence $H_i(C'') = F[\tilde{a}^i]$ for $i = 2, 3$. We fix the basis $[\tilde{a}^2], [\tilde{a}^3]$ in $H_*(C'')$. Using the exact homology sequence of the pair (C, C') we obtain that $H_i(C') = 0$ for $i \neq 1, 2$ and $H_i(C') = F[\partial \tilde{a}^{i+1}]$ for $i = 1, 2$. We fix the basis $[\partial \tilde{a}^2], [\partial \tilde{a}^3]$ in $H_*(C')$. A direct computation shows that $\check{\tau}(C'') = \tau(C' \subset C) = -1$. Hence

$$\begin{aligned} \tau(C) = \check{\tau}(C) &= (-1)^{\nu(C,C')} \check{\tau}(C')\, \check{\tau}(C'')\, \tau(C' \subset C) \\ &= (-1)^{\nu(C,C')} \check{\tau}(C'; [\partial \tilde{a}^2], [\partial \tilde{a}^3]) \end{aligned} \tag{6.2.b}$$

where we inserted $[\partial \tilde{a}^2], [\partial \tilde{a}^3]$ to keep track of the fixed basis in $H_*(C') = H_*^{\varphi}(E)$.

The same argument as in Section 5.3 shows that

$$\check{\tau}_0(C_*(M; \mathbf{R})) = (-1)^{\nu(C,C')}\, \check{\tau}_0(C_*(E; \mathbf{R}))$$

where the torsions are taken with respect to the bases of chains determined by $\tilde{e}, a^2, a^3$ and bases in homology determining the homology orientations ω, ω^M. Multiplying by (6.2.b) we obtain,

$$\tau^{\varphi^M}(M, e^M, \omega^M) = \check{\tau}_0(C_*(M; \mathbf{R}))\, \tau(C)$$

$$= \check{\tau}_0(C_*(E; \mathbf{R}))\, \check{\tau}(C'; [\partial \tilde{a}^2], [\partial \tilde{a}^3]) = \tau^{\varphi}(E, e, \omega; [\partial \tilde{a}^2], [\partial \tilde{a}^3]).$$

It remains to give an interpretation of $[\partial \tilde{a}^2], [\partial \tilde{a}^3]$ in terms of $\hat{E}$. The complex C' can be computed from $\hat{E}$. Indeed, a natural projection from the maximal abelian covering $\tilde{E}$ of E to $\hat{E}$ induces the equalities

$$C' = C_*^{\varphi}(E) = F \otimes_{\mathbf{Z}[H_1(E)]} C_*(\tilde{E}) = F \otimes_{\mathbf{Z}[G]} C_*(\hat{E})$$

where $G = H_1(E)/(H_1(E) \cap \varphi^{-1}(1)) = \varphi(H_1(E))$ is the group of covering transformations of $\hat{E}$. The assumption $\varphi(H_1(T)) = 1$ implies that T lifts to a torus $\hat{T} \subset \partial \hat{E}$. We provide $\hat{T}$ with the orientation induced by the one on $T \subset \partial E$. Since the projection $\hat{T} \to T$ is a homeomorphism, the curve α lifts to a curve $\hat{\alpha}$ on $\hat{T}$. Then $\hat{\alpha}$ and $\hat{T}$ represent certain homology classes $[\hat{\alpha}] \in H_1^\varphi(E), [\hat{T}] \in H_2^\varphi(E)$.

The covering $\hat{E} \to E$ extends to a covering $\hat{M} \to M$ with the same group of covering transformations G. The torus $\hat{T}$ bounds in $\hat{M}$ a solid torus, $\hat{\mathrm{Z}}$, projecting homeomorphically onto Z, and $\hat{M} = \hat{E} \cup_{g \in G} g\hat{\mathrm{Z}}$. There is a covering $\tilde{M} \to \hat{M}$ mapping $\tilde{a}^2$ onto a meridional disc of $\hat{\mathrm{Z}}$ bounded by $\hat{\alpha}$ and mapping $\tilde{a}^3$ onto a 3-cell filling in $\hat{\mathrm{Z}}$. Then $[\partial \tilde{a}^2] = [\hat{\alpha}] \in H_1^\varphi(E)$ and $[\partial \tilde{a}^3] = [\partial \hat{\mathrm{Z}}] = -\varepsilon\, [\hat{T}] \in H_2^\varphi(E)$. Therefore

$$\tau^{\varphi^M}(M, e^M, \omega^M) = \tau^\varphi(E, e, \omega; [\partial \tilde{a}^2], [\partial \tilde{a}^3]) = -\varepsilon\, \tau^\varphi(E, e, \omega; [\hat{\alpha}], [\hat{T}]).$$

6.3. Remark. The fact that the torsion $\tau^\varphi(E, e, \omega; [\hat{\alpha}], [\hat{T}])$ in Lemma 6.2 does not depend on the choice of $\hat{T}$ follows directly from the obvious equality $\tau^\varphi(E, e, \omega; g[\hat{\alpha}], g[\hat{T}]) = \tau^\varphi(E, e, \omega; [\hat{\alpha}], [\hat{T}])$ where $g \in G = \varphi(H_1(E))$.

6.4. Lemma. *Let E be a compact 3-manifold such that ∂E consists of tori. Let T be a component of ∂E. Let F be a field and $\varphi : \mathbf{Z}[H_1(E)] \to F$ be a ring homomorphism such that $\varphi(H_1(T)) = 1$. Let $\alpha_1, \alpha_2, \alpha_3$ be oriented non-contractible simple closed curves on T such that α_1 is homological to $\alpha_2\alpha_3$. Fix $\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1$ and set $M_i = M_{E,\alpha_r,\varepsilon_r}$ for $r = 1, 2, 3$. Then for any $e \in \mathrm{Eul}(E)$ and any homology orientation ω of E,*

$$\begin{aligned} &\varepsilon_1\, \tau^{\varphi^{M_1}}(M_1, e^{M_1}, \omega^{M_1}) \\ &= \varepsilon_2\, \tau^{\varphi^{M_2}}(M_2, e^{M_2}, \omega^{M_2}) + \varepsilon_3\, \tau^{\varphi^{M_3}}(M_3, e^{M_3}, \omega^{M_3}). \qquad (6.4.a) \end{aligned}$$

Proof. Let $\hat{E} \to E$ be the same covering of E as in Lemma 6.2 with group of covering transformations $G = \varphi(H_1(E))$. Let $\hat{T} \subset \hat{E}$ be a lift of T and $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3$ be the lifts of $\alpha_1, \alpha_2, \alpha_3$ to $\hat{T}$. Consider the elements $[\hat{\alpha}_1], [\hat{\alpha}_2], [\hat{\alpha}_3]$ of $H_1^\varphi(E) = H_1(F \otimes_{\mathbf{Z}[G]} C_*(\hat{E}))$ represented by $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3$, respectively. The assumptions of the lemma imply that $[\hat{\alpha}_1] = [\hat{\alpha}_2] + [\hat{\alpha}_3]$.

If all three torsions entering (6.4.a) are equal to 0 then (6.4.a) is obvious. Assume from now on that at least one of these three torsions is non-zero. Then the proof of Lemma 6.2 shows that $H_i^\varphi(E) = 0$ for $i \neq 1, 2$ and $H_1^\varphi(E), H_2^\varphi(E)$ are 1-dimensional vector spaces over F. Moreover, $H_2^\varphi(E) = F\, [\hat{T}]$. It is easy to see that $[\hat{\alpha}_r] \neq 0$ iff $H_*^{\varphi^{M_r}}(M_r) = 0$ iff $\tau^{\varphi^{M_r}}(M_r, e^{M_r}, \omega^{M_r}) \neq 0$. By assumption, at least one of the classes $[\hat{\alpha}_1], [\hat{\alpha}_2], [\hat{\alpha}_3]$ is non-zero.

If all three classes $[\hat{\alpha}_1], [\hat{\alpha}_2], [\hat{\alpha}_3]$ are non-zero then by Lemma 6.2

$$\varepsilon_1\, \tau^{\varphi^{M_1}}(M_1, e^{M_1}, \omega^{M_1}) = -\tau^\varphi(E, e, \omega; [\hat{\alpha}_1], [\hat{T}])$$

$$= -\tau^{\varphi}(E, e, \omega; [\hat{\alpha}_2], [\hat{T}]) - \tau^{\varphi}(E, e, \omega; [\hat{\alpha}_3], [\hat{T}])$$

$$= \varepsilon_2 \, \tau^{\varphi^{M_2}}(M_2, e^{M_2}, \omega^{M_2}) + \varepsilon_3 \, \tau^{\varphi^{M_3}}(M_3, e^{M_3}, \omega^{M_3})$$

where the second equality follows from the formula $[\hat{\alpha}_1] = [\hat{\alpha}_2] + [\hat{\alpha}_3]$. If $[\hat{\alpha}_1] = 0$ then $[\hat{\alpha}_2] = -[\hat{\alpha}_3] \neq 0$. Then $\tau^{\varphi^{M_1}}(M_1, e^{M_1}, \omega^{M_1}) = 0$ and

$$\varepsilon_2 \, \tau^{\varphi^{M_2}}(M_2, e^{M_2}, \omega^{M_2}) = -\tau^{\varphi}(E, e, \omega; [\hat{\alpha}_2], [\hat{T}])$$

$$= \tau^{\varphi}(E, e, \omega; [\hat{\alpha}_3], [\hat{T}]) = -\varepsilon_3 \, \tau^{\varphi^{M_3}}(M_3, e^{M_3}, \omega^{M_3})$$

which proves (6.4.a). The cases where $[\hat{\alpha}_2] = 0$ or $[\hat{\alpha}_3] = 0$ are similar.

7. The torsion τ

We discuss the "maximal abelian" torsion τ introduced in [Tu2], see also [Tu5], [Tu7].

7.1. Algebraic preliminaries. For a unital commutative ring K we denote by $Q(K)$ its *classical ring of fractions*, i.e., the localisation of K by the multiplicative system of all non-zerodivisors. The natural map $K \to Q(K)$ is injective.

For a finitely generated abelian group H, set $Q(H) = Q(\mathbf{Z}[H]) = Q(\mathbf{Q}[H])$. This commutative ring splits (uniquely) as a direct sum of fields. This is obvious in the case of finite H since then $\mathbf{Q}[H]$ is a finite sum of cyclotomic fields so that $Q(H) = \mathbf{Q}[H]$. In the general case such a splitting comes from a splitting of H as a direct sum of $\operatorname{Tors} H$ and the free abelian group $G = H/\operatorname{Tors} H$. The ring $\mathbf{Q}[\operatorname{Tors} H]$ splits as a sum of cyclotomic fields $\{K_r\}_r$. Hence

$$\mathbf{Q}[H] = (\mathbf{Q}[\operatorname{Tors} H])[G] = \bigoplus_r K_r[G].$$

Each ring $K_r[G]$ is a domain and therefore

$$Q(H) = \bigoplus_r F_r \qquad (7.1.a)$$

where $F_r = Q(K_r[G])$ is the field of fractions of $K_r[G]$. A useful fact: for any $h \in H$ of infinite order, the element $h - 1 \in \mathbf{Z}[H]$ is a non-zerodivisor and therefore is invertible in $Q(H)$.

One of the summands in (7.1.a) is the field $Q(H/\operatorname{Tors} H)$. The projection $Q(H) \to Q(H/\operatorname{Tors} H)$ is induced by the projection $H \to H/\operatorname{Tors} H$. The inclusion $Q(H/\operatorname{Tors} H) \hookrightarrow Q(H)$ maps $1 \in Q(H/\operatorname{Tors} H)$ into $|\operatorname{Tors} H|^{-1} \sum_{h \in \operatorname{Tors} H} h$.

Consider an increasing filtration

$$(\mathbf{Z}[H])_0 = \mathbf{Z}[H] \subset (\mathbf{Z}[H])_1 \subset (\mathbf{Z}[H])_2 \subset ... \subset Q(H) \qquad (7.1.b)$$

where $(\mathbf{Z}[H])_k$ is the set of $q \in Q(H)$ such that for any $h_1, ..., h_k \in H$ we have $q \prod_{i=1}^k (h_i - 1) \in \mathbf{Z}[H]$. We shall need the following easy fact: any ring homomorphism φ from $\mathbf{Z}[H]$ to a field F such that $\varphi(H) \neq 1$ uniquely extends to a ring homomorphism $\varphi_\# : \cup_k (\mathbf{Z}[H])_k \to F$. If $\operatorname{rank} H \geq 2$ then the filtration (7.1.b) is trivial, i.e., $(\mathbf{Z}[H])_k = \mathbf{Z}[H]$ for all k (see [Tu5, Lemma 4.1.1]) and $\varphi_\# = \varphi : \mathbf{Z}[H] \to F$.

7.2. Definition of τ. Let X be a finite connected CW-space. Set $H = H_1(X)$ and denote by φ_r the composition of the inclusion $\mathbf{Z}[H] \hookrightarrow Q(H)$ and the projection $Q(H) \to F_r$ on the r-th term in the splitting (7.1.a). By Sect. 4.2, for any $e \in \operatorname{Eul}(X)$ and any homology orientation ω of X, we have a torsion $\tau^{\varphi_r}(X, e, \omega) \in F_r$. Set

$$\tau(X, e, \omega) = \bigoplus_r \tau^{\varphi_r}(X, e, \omega) \ \in \ \bigoplus_r F_r = Q(H).$$

This is a well defined element of $Q(H)$. For any $h \in H$, $\tau(X, he, \omega) = h\,\tau(X, e, \omega)$.

7.3. The torsion τ for 3-manifolds. We shall need three basic lemmas concerning the torsion τ for 3-manifolds.

7.3.1. Lemma. *Let M be a compact connected orientable 3-manifold whose boundary is empty or consists of tori. Let $e \in \operatorname{Eul}(M)$ and ω be a homology orientation of M. Set $H = H_1(M)$. If $b_1(M) \geq 2$, then $\tau(M, e, \omega) \in \mathbf{Z}[H]$. If $b_1(M) = 1$ and $\partial M \neq \emptyset$, then $\tau(M, e, \omega) \in (\mathbf{Z}[H])_1$. If $b_1(M) = 1$ and $\partial M = \emptyset$, then $\tau(M, e, \omega) \in (\mathbf{Z}[H])_2$.*

Lemma 7.3.1 follows from the results of [Tu5, Section 4]. Although we shall not need it, note that the inclusion $\tau(M, e, \omega) \in Q(H)$ can be improved also in the case $b_1(M) = 0$: In this case $\tau(M, e, \omega) \in |\operatorname{Tors} H|^{-1} \mathbf{Z}[H]$.

7.3.2. Lemma. *Let under the conditions of Lemma 7.3.1, φ be a ring homomorphism from $\mathbf{Z}[H]$ to a field F such that $\varphi(H) \neq 1$ and $\operatorname{char} F = 0$. Then $\tau^\varphi(M, e, \omega) = \varphi_\#(\tau(M, e, \omega))$.*

In the case $\tau^\varphi(M, e, \omega) \neq 0$ this lemma is contained in [Tu7, Theorem 13.3] in a general setting of CW-spaces of any dimension. It remains only to show that if $\tau^\varphi(M, e, \omega) = 0$, i.e., if $H_*^\varphi(M) \neq 0$, then $\varphi_\#(\tau(M, e, \omega)) = 0$. This easily follows from the computations in [Tu5, Sect. 4.1.2] and the assumption $\varphi(H) \neq 1$.

7.3.3. Lemma. *Let E be a compact connected orientable 3-manifold such that ∂E consists of tori and $b_1(E) \geq 2$. Let M be a 3-manifold with $b_1(M) \geq 1$ obtained by gluing m directed solid tori to E and let $h_1, ..., h_m \in H_1(M)$ be the homology classes of the core circles of these solid tori. Let $\operatorname{in} : \mathbf{Z}[H_1(E)] \to \mathbf{Z}[H_1(M)]$ be the inclusion homomorphism. Then for any $e \in \operatorname{Eul}(E)$ and any homology orientation*

ω *of* E,

$$\mathrm{in}(\tau(E,e,\omega)) = \prod_{i=1}^{m} (h_i - 1)\, \tau(M, e^M, \omega^M). \qquad (7.3.a)$$

Proof. Consider the splitting $Q(H_1(M)) = \bigoplus_r F_r$ into a direct sum of fields. Denote the projection $Q(H_1(M)) \to F_r$ by p_r. It suffices to prove that for all r,

$$p_r(\mathrm{in}(\tau(E,e,\omega))) = \prod_{i=1}^{m} (p_r(h_i) - 1)\, p_r(\tau(M, e^M, \omega^M)).$$

Let μ be the inclusion $\mathbf{Z}[H_1(M)] \hookrightarrow Q(H_1(M))$. Applying Lemmas 7.3.2 twice and Lemma 5.1 inductively m times we obtain that

$$p_r(\mathrm{in}(\tau(E,e,\omega))) = (p_r\,\mu\,\mathrm{in})(\tau(E,e,\omega)) = (p_r\,\mu\,\mathrm{in})_\#(\tau(E,e,\omega)) = \tau^{p_r \mu\, \mathrm{in}}(E,e,\omega)$$

$$= \prod_{i=1}^{m} (p_r(h_i) - 1)\, \tau^{p_r \mu}(M, e^M, \omega^M) = \prod_{i=1}^{m} (p_r(h_i) - 1)\, p_r(\tau(M, e^M, \omega^M)).$$

8. The Alexander-Conway function and derived invariants

8.1. The Alexander-Conway function. Let $L = L_1 \cup ... \cup L_m$ be an ordered oriented link in an oriented 3-dimensional integral homology sphere N. The Alexander-Conway function ∇_L of L is a rational function on m variables $t_1, ..., t_m$ with integer coefficients. Considered up to sign, this function is equivalent to the m-variable Alexander polynomial of L. For links in S^3, the function ∇_L was introduced by Conway [Co] (see also [Ha]); it was extended to links in homology spheres in [Tu3]. We recall here the definition of ∇_L following [Tu3, Sect. 4].

Let E be the exterior of L. The group $H = H_1(E)$ is a free abelian group with m meridional generators $t_1, ..., t_m$. We have $\mathbf{Z}[H] = \mathbf{Z}[t_1^{\pm 1}, ..., t_m^{\pm 1}]$ (the ring of of Laurent polynomials) and $Q(H) = \mathbf{Z}(t_1, ..., t_m)$ (the ring of rational functions on $t_1, ..., t_m$ with integer coefficients). Denote by μ the inclusion $\mathbf{Z}[H] \hookrightarrow Q(H)$. Choose any Euler structure e on E and set

$$A_e = A_e(t_1, ..., t_m) = \tau^\mu(E, e, \omega_L) \in Q(H) = \mathbf{Z}(t_1, ..., t_m).$$

It is known (see for instance [Tu3, Sect. 1.7]) that $\overline{A}_e = (-1)^m t_1^{\nu_1} ... t_m^{\nu_m} A_e$ where the overbar denotes the involution in $Q(H)$ sending each t_i to t_i^{-1} and $(\nu_1, ..., \nu_m) \in \mathbf{Z}^m$ is a charge on L in the sense of Sect. 1.6. Then

$$\nabla_L = -t_1^{\nu_1} ... t_m^{\nu_m} A_e(t_1^2, ..., t_m^2) \in \mathbf{Z}(t_1, ..., t_m). \qquad (8.1.a)$$

Clearly, $\overline{\nabla}_L = (-1)^m \nabla_L$. By (4.2.a), ∇_L does not depend on the choice of e.

We introduce a version of ∇_L depending on a charge $k=(k_1,...,k_m)\in \mathbf{Z}^m$ on L. Set

$$\nabla(L,k)=-t_1^{k_1/2}...t_m^{k_m/2}\,\nabla_L(t_1^{1/2},...,t_m^{1/2}).$$

Note that although the factors on the right-hand side can lie in $\mathbf{Z}(t_1^{1/2},...,t_m^{1/2})$, their product $\nabla(L,k)$ belongs to $\mathbf{Z}(t_1,...,t_m)$. We claim that

$$\nabla(L,k)=\tau(E,e_k,\omega_L). \tag{8.1.b}$$

Indeed, the splitting (7.1.a) consists here of only one summand and therefore $\tau(E,e_k,\omega_L)=\tau^\mu(E,e_k,\omega_L)=A_{e_k}$. By (4.6.a),

$$t_1^{\nu_1}...t_m^{\nu_m}=\pm\overline{A}_{e_k}/A_{e_k}=\pm c(e_k)^{-1}=\pm t_1^{-k_1}...t_m^{-k_m}$$

so that $\nu_i=-k_i$ for $i=1,...,m$. Substituting this in (8.1.a) with $e=e_k$ we obtain a formula equivalent to (8.1.b).

To end this subsection note that for $m\geq 2$ both ∇_L and $\nabla(L,k)$ are Laurent polynomials, i.e., belong to $\mathbf{Z}[t_1^{\pm1},...,t_m^{\pm1}]$. If $m=1$ then ∇_L and $\nabla(L,k)$ are rational functions on $t=t_1$ which can be computed from the Alexander polynomial of $L=L_1$. Let $\Delta_L(t)\in\mathbf{Z}[t^{\pm1}]$ be the Alexander polynomial of L normalized (and symmetrized) in the canonical way so that $\Delta_L(t^{-1})=\Delta_L(t)$ and $\Delta_L(1)=1$. Then

$$\nabla_L=(t-t^{-1})^{-1}\Delta_L(t^2),\quad \nabla(L,k)=-t^{(k+1)/2}(t-1)^{-1}\Delta_L(t) \tag{8.1.c}$$

where $k=k_1\in\mathbf{Z}$ is the charge of L.

8.2. An invariant of sublinks. Let $L=L_1\cup...\cup L_m$ be an m-component ordered oriented link in an oriented 3-dimensional integral homology sphere N. Let $L^I=\{L_i\}_{i\in I}$ be a sublink of L determined by a proper subset $I\subset\{1,...,m\}$. In this subsection we associate with (L,L^I) an abelian group $H=H(L,L^I)$. For any charge k on L (see Sect. 1.6), we define an element $\nabla(L,L^I,k)\in Q(H)$.

Set $|I|=\mathrm{card}(I)$. The group $H=H(L,L^I)$ is defined as the quotient of the free (multiplicative) abelian group of rank m with generators $t_1,...,t_m$ modulo the following $|I|$ relations numerated by $i\in I$:

$$\prod_{j\neq i}(t_jt_i^{-1})^{lk(L_i,L_j)}=1 \tag{8.2.a}$$

where the product goes over all $j\in\{1,...,m\}\backslash\{i\}$. The ring $\mathbf{Z}[H]$ can be identified with the ring of polynomials on m commuting invertible variables $t_1,...,t_m$ subject to (8.2.a). There is a group homomorphism from H to the infinite cyclic group sending $t_1,...,t_m$ to the same generator. Therefore each t_j has an infinite order in H and t_j-1 is a non-zerodivisor in $\mathbf{Z}[H]$ invertible in $Q(H)$. Set

$$\nabla(L,L^I,k)=\left(\prod_{i\in I}(t_i-1)^{-1}\right)\nabla(L,k)\in Q(H). \tag{8.2.b}$$

Here $\nabla(L,k)$ is viewed as a rational function on $t_1,...,t_m$ representing an element of $Q(H)$. In particular, if $I=\emptyset$, then $\nabla(L,L^I,k)=\nabla(L,k)$.

The group $H=H(L,L^I)$ and the element $\nabla(L,L^I,k)\in Q(H)$ can be computed from the linking numbers of the components of L and $\nabla(L,k)$. We give a geometric interpretation of H and $\nabla(L,L^I,k)$. Let E be the exterior of L in N. Let $P=P(L,L^I)$ be the 3-manifold obtained from E by gluing directed solid tori $\{\mathrm{Z}_i\}_{i\in I}$ such that the longitude of L_i with $i\in I$ determined by the framing number $-lk(L_i,L\backslash L_i)=-\sum_{j\neq i} lk(L_i,L_j)$ bounds a meridional disc in Z_i. The core of Z_i is oriented as in Sect. 1.7. The boundary of P consists of $m-|I|\geq 1$ tori corresponding to the components of $L\backslash L^I$. Identifying $t_1,...,t_m$ with the meridional generators of $H_1(E)$ we obtain an identification $H_1(P)=H$. We claim that

$$\nabla(L,L^I,k)=\tau(P,e_k^P,\omega_L^P) \tag{8.2.c}$$

where e_k is the Euler structure on E determined by k, $e_k^P=(e_k)^P$ is the induced Euler structure on P (cf. Sect. 1.5–1.7), ω_L is the homology orientation of E determined by the orientation of L, and $\omega_L^P=(\omega_L)^P$ is the induced homology orientation on P (cf. Sect. 3.1, 3.2). Indeed, by (7.3.a) and (8.1.b)

$$\left(\prod_{i\in I}(t_i-1)\right)\tau(P,e_k^P,\omega_L^P)=\tau(E,e_k,\omega_L)=\nabla(L,k).$$

This and (8.2.b) yield (8.2.c). This interpretation of $\nabla(L,L^I,k)$ implies that if $\operatorname{rank} H\geq 2$ then $\nabla(L,L^I,k)\in\mathbf{Z}[H]$. If $\operatorname{rank} H=1$ then $\nabla(L,L^I,k)\in(\mathbf{Z}[H])_1$.

9. A surgery formula for φ-torsions

9.1. Setting and notation. Let M be a closed 3-manifold obtained by surgery on a framed oriented link $L=L_1\cup...\cup L_m$ in an oriented 3-dimensional integral homology sphere N. Let F be a field of characteristic 0. Fix a charge k of L and consider a ring homomorphism $\varphi^M:\mathbf{Z}[H_1(M)]\to F$ such that $\varphi^M(H_1(M))\neq 1$. We shall give a surgery formula for the torsion $\tau^{\varphi^M}(M,e_k^M,\omega_L^M)\in F$ where $e_k^M=(e_k)^M\in\operatorname{vect}(M)$ is induced by the Euler structure e_k on the exterior E of L and $\omega_L^M=(\omega_L)^M$ is the homology orientation of M induced by the homology orientation ω_L of E (see Sect. 3.1).

Recall the notation $\overline{I}=\{1,...,m\}\backslash I$ for a set $I\subset\{1,...,m\}$. Denote by $\ell^I(L)=(\ell^I_{i,j}(L))$ the symmetric square matrix whose rows and columns are numerated by $i,j\in I$ and whose entries are given by

$$\ell^I_{i,j}(L)=\begin{cases} lk(L_i,L_j),\ \ if\ \ i\neq j,\\ lk(L_i,L_i)+lk(L_i,L^{\overline{I}})=lk(L_i,L_i)+\sum_{l\in\overline{I}} lk(L_i,L_l),\ \ if\ \ i=j.\end{cases}$$

Here $lk(L_i,L_i)$ is the framing number of L_i.

Denote by φ the composition of φ^M with the ring homomorphism $\mathbf{Z}[H_1(E)] \to \mathbf{Z}[H_1(M)]$ induced by the inclusion $E \subset M$. Let $t_1, ..., t_m \in H_1(E)$ be the homology classes of the meridians of $L_1, ..., L_m$. Set

$$I(\varphi) = \{i \,|\, \varphi(t_i) = 1\} \subset \{1, ..., m\}.$$

For any $i \in I(\varphi)$, the homomorphism φ maps the meridian t_i and the longitude of L_i to $1 \in F$. This implies that for a set $I \subset \{1, ..., m\}$, not contained in $I(\varphi)$, the formula $\{t_i \mapsto \varphi(t_i) \in F\}_{i \in I}$ defines a ring homomorphism $\mathbf{Z}[H(L^I, L^{I \cap I(\varphi)})] \to F$ (cf. Sect. 8.2). It is denoted φ^I. Recall the notation $|I| = \mathrm{card}(I)$.

9.2. Theorem. *Under the assumptions of Sect. 9.1,*

$$\tau^{\varphi^M}(M, e_k^M, \omega_L^M) \tag{9.2.a}$$

$$= \left(\prod_{i \in \overline{I(\varphi)}} (\varphi(t_i) - 1)^{-1} \right) \sum_{I \subset I(\varphi)} (-1)^{|I|} \det(\ell^I(L))\, \varphi_\#^{\overline{I}}(\nabla(L^{\overline{I}}, L^{\overline{I} \cap I(\varphi)}, k^{\overline{I}})).$$

Formula (9.2.a) allows to compute the φ-torsion of M in terms of the Alexander-Conway polynomials of L and its sublinks, and the linking and framing numbers of the components of L. A curious feature of this formula is a framing-charge separation: the factor $\det(\ell^I)$ does not depend on the charge while the factor $\varphi_\#^{\overline{I}}(\nabla(L^{\overline{I}}, L^{\overline{I} \cap I(\varphi)}, k^{\overline{I}}))$ does not depend on the framing. The product $\prod_{i \in \overline{I(\varphi)}} (\varphi(t_i) - 1)^{-1}$ depends neither on the charge nor on the framing.

The terms on the right-hand side of (9.2.a) corresponding to proper subsets $I \subset I(\varphi)$ can be slightly simplified. Observe that

$$\mathrm{rank}\, H(L^{\overline{I}}, L^{\overline{I} \cap I(\varphi)}) \geq |\overline{I} \backslash I(\varphi)| = |\overline{I(\varphi)}| + |I(\varphi) \backslash I|.$$

The assumption $\varphi(H_1(E)) \neq 1$ implies that $|\overline{I(\varphi)}| \geq 1$. Therefore for a proper subset $I \subset I(\varphi)$ we have $\mathrm{rank}\, H(L^{\overline{I}}, L^{\overline{I} \cap I(\varphi)}) \geq 2$ and

$$\varphi_\#^{\overline{I}}(\nabla(L^{\overline{I}}, L^{\overline{I} \cap I(\varphi)}, k^{\overline{I}})) = \varphi^{\overline{I}}(\nabla(L^{\overline{I}}, L^{\overline{I} \cap I(\varphi)}, k^{\overline{I}})).$$

The term corresponding to $I = I(\varphi)$ equals

$$\prod_{i \in \overline{I(\varphi)}} (\varphi(t_i) - 1)^{-1} (-1)^{|I(\varphi)|} \det(\ell^{I(\varphi)}(L))\, \varphi_\#^{\overline{I(\varphi)}}(\nabla(L^{\overline{I(\varphi)}}, k^{\overline{I(\varphi)}})).$$

We can replace here $\varphi_\#^{\overline{I(\varphi)}}$ by $\varphi^{\overline{I(\varphi)}}$ provided $|\overline{I(\varphi)}| \geq 2$.

If $I(\varphi) = \emptyset$, i.e., if $\varphi(t_i) \neq 1$ for all $i = 1, ..., m$, then (9.2.a) simplifies to

$$\tau^{\varphi^M}(M, e_k^M, \omega_L^M) = \prod_{i=1}^{m} (\varphi(t_i) - 1)^{-1}\, \varphi_\#(\nabla(L, k)). \tag{9.2.b}$$

This can be obtained by applying (5.1.a) m times followed by Lemma 7.3.2 and (8.1.b):

$$\tau^{\varphi^M}(M, e_k^M, \omega_L^M) = \prod_{i=1}^{m} (\varphi(t_i) - 1)^{-1}\, \tau^{\varphi}(E, e_k, \omega_L)$$

$$= \prod_{i=1}^{m} (\varphi(t_i) - 1)^{-1}\, \varphi_{\#}(\tau(E, e_k, \omega_L)) = \prod_{i=1}^{m} (\varphi(t_i) - 1)^{-1}\, \varphi_{\#}(\nabla(L, k)).$$

Proof of Theorem 9.2. The proof goes by induction on m. If $m = 1$ then $I(\varphi) = \emptyset$ and (9.2.a) reduces to (9.2.b) proven above. Assume that for links with $< m$ components, Theorem 9.2 is true and prove it for a link L with m components.

Consider first the case where for any $i \in I(\varphi)$ the framing number $lk(L_i, L_i)$ is equal to $-\sum_{l \neq i} lk(L_i, L_l)$ (the framing numbers of $\{L_i \,|\, i \in \overline{I(\varphi)}\}$ may be arbitrary). Then for any non-void set $I \subset I(\varphi)$, the sum of the columns of the matrix ℓ^I is 0 so that $\det(\ell^I(L)) = 0$. Thus, the right-hand side of (9.2.a) contains only one possibly non-zero term corresponding to $I = \emptyset$. By (8.2.c), this term equals

$$\prod_{i \in \overline{I(\varphi)}} (\varphi(t_i) - 1)^{-1}\, \psi_{\#}(\nabla(L, L^{I(\varphi)}, k)) = \prod_{i \in \overline{I(\varphi)}} (\varphi(t_i) - 1)^{-1}\, \psi_{\#}(\tau(P, e_k^P, \omega_L^P))$$

where $P = P(L, L^{I(\varphi)})$ is the 3-manifold defined as in Sect. 8.2 and $\psi = \varphi^{\overline{\emptyset}}$ is the ring homomorphism $\mathbf{Z}[H_1(P)] = \mathbf{Z}[H(L, L^{I(\varphi)})] \to F$ induced by φ. By Lemma 7.3.2,

$$\prod_{i \in \overline{I(\varphi)}} (\varphi(t_i) - 1)^{-1}\, \psi_{\#}(\tau(P, e_k^P, \omega_L^P)) = \prod_{i \in \overline{I(\varphi)}} (\varphi(t_i) - 1)^{-1}\, \tau^{\psi}(P, e_k^P, \omega_L^P).$$

The manifold M can be obtained from P by gluing directed solid tori corresponding to the components of $L^{\overline{I(\varphi)}}$. Applying (5.1.a) inductively and using Lemma 3.3, we obtain

$$\prod_{i \in \overline{I(\varphi)}} (\varphi(t_i) - 1)^{-1}\, \tau^{\psi}(P, e_k^P, \omega_L^P) = \tau^{\varphi^M}(M, (e_k^P)^M, (\omega_L^P)^M) = \tau^{\varphi^M}(M, e_k^M, \omega_L^M).$$

This yields (9.2.a) in this case.

Up to the rest of the proof, we denote the left (resp. right) hand side of (9.2.a) by $\langle L, \varphi, k \rangle_1$ (resp. $\langle L, \varphi, k \rangle_2$). We claim that for a framed link L' obtained from L by one negative twist of the framing of a component of $L^{I(\varphi)} \subset L$,

$$\langle L, \varphi, k \rangle_1 - \langle L, \varphi, k \rangle_2 = \langle L', \varphi, k \rangle_1 - \langle L', \varphi, k \rangle_2. \qquad (9.2.c)$$

Hence, formula (9.2.a) holds for L if and only if it holds for L'. Together with the result of the previous paragraph this will imply the claim of the theorem.

Now we check (9.2.c). Assume for concreteness that $m \in I(\varphi)$ and that $L' = L_1 \cup ... \cup L_{m-1} \cup L'_m$ is obtained from L by one negative twist of the framing of L_m. Consider the closed 3-manifold, W, obtained by surgery on N along the framed link $L^{\overline{m}} = L_1 \cup ... \cup L_{m-1}$. The framed oriented knots L_m, L'_m lie in W and differ only by the framing. These knots have the same exterior in W, which we denote by V. It is clear that ∂V is a 2-torus bounding a regular neighborhood of L_m in W which can be identified with a regular neighborhood of L_m in N. Let $\alpha_1 \subset \partial V$ (resp. $\alpha_2 \subset \partial V$) be a longitude of L_m (resp. of L'_m) determined by the framing. Let $\alpha_3 \subset \partial V$ be a meridian of L_m. By assumption, α_1 is homological to $\alpha_2\alpha_3$ in ∂V. In the notation of Sect. 6.1 we have $M = M_{V,\alpha_1,-1}$ (the sign -1 appears because the orientation of M induced from the one in N is *opposite* to the product orientation in the directed solid torus $M \backslash V$; see Sect. 1.7 and 6.1 for the distinguished orientations of the core and meridional disc). Set $M' = M_{V,\alpha_2,-1}$. Observe that $M_{V,\alpha_3,+1} = W$. As in Sect. 1.7, the manifold V is obtained from the exterior E of L by gluing $m-1$ directed solid tori. Set $e = (e_k)^V \in \mathrm{vect}(V)$ and provide V with homology orientation $\omega = (\omega_L)^V$. Denote by φ^V the ring homomorphism $\mathbf{Z}[H_1(V)] \to F$ induced by φ; this homomorphism is the composition of the inclusion homomorphism $\mathbf{Z}[H_1(V)] \to \mathbf{Z}[H_1(M)]$ with φ^M. By Lemma 6.4,

$$\tau^{(\varphi^V)^M}(M, e^M, \omega^M) = \tau^{(\varphi^V)^{M'}}(M', e^{M'}, \omega^{M'}) - \tau^{(\varphi^V)^W}(W, e^W, \omega^W). \quad (9.2.d)$$

By definition of φ^V, we have $(\varphi^V)^M = \varphi^M$. By definition of e_k^M, we have $e^M = e_k^M$. By Lemma 3.3, $\omega^M = ((\omega_L)^V)^M = \omega_L^M$. Thus the left-hand side of (9.2.d) equals $\langle L, \varphi, k\rangle_1$. The manifold M' is obtained by surgery on N along L' and similarly

$$\tau^{(\varphi^V)^{M'}}(M', e^{M'}, \omega^{M'}) = \langle L', \varphi, k\rangle_1.$$

Consider now the exterior $E^{\overline{m}}$ of $L^{\overline{m}} = L_1 \cup ... \cup L_{m-1}$ in N, the ring homomorphism $\psi : \mathbf{Z}[H_1(E^{\overline{m}})] \to F$ induced by φ, the charge $k^{\overline{m}}$ on $L^{\overline{m}}$ induced by k as in Sect. 1.6 and the homology orientation $\omega_{L^{\overline{m}}}$ of $E^{\overline{m}}$ nduced by the orientation of $L^{\overline{m}}$. We claim that

$$\tau^{(\varphi^V)^W}(W, e^W, \omega^W) = \langle L^{\overline{m}}, \psi, k^{\overline{m}}\rangle_1. \quad (9.2.e)$$

Indeed, the manifold W is obtained by surgery on N along $L^{\overline{m}}$ and hence by gluing directed solid tori to $E^{\overline{m}}$. It follows from definitions that $(\varphi^V)^W = \psi^W$ (both homomorphisms map a meridional generator $t_i \in H_1(W)$ with $i = 1, ..., m-1$ to $\varphi(t_i)$). The remarks of Sect. 1.6 imply that $(e_{k^{\overline{m}}})^W = ((e_k)^V)^W = e^W$. Lemmas 3.3 and 3.4 imply that $(\omega_{L^{\overline{m}}})^W = ((\omega_L)^V)^W = \omega^W$. Hence

$$\tau^{(\varphi^V)^W}(W, e^W, \omega^W) = \tau^{\psi^W}(W, (e_{k^{\overline{m}}})^W, (\omega_{L^{\overline{m}}})^W) = \langle L^{\overline{m}}, \psi, k^{\overline{m}}\rangle_1.$$

Now formula (9.2.d) can be rewritten as

$$\langle L, \varphi, k\rangle_1 - \langle L', \varphi, k\rangle_1 = -\langle L^{\overline{m}}, \psi, k^{\overline{m}}\rangle_1. \quad (9.2.e)$$

We claim that similarly

$$\langle L, \varphi, k\rangle_2 - \langle L', \varphi, k\rangle_2 = -\langle L^{\overline{m}}, \psi, k^{\overline{m}}\rangle_2. \qquad (9.2.f)$$

Indeed, it is clear that $I(\psi) = I(\varphi)\backslash\{m\}$ so that the complement of $I(\varphi)$ in $\{1, ..., m\}$ coincides with the complement of $I(\psi)$ in $\{1, ..., m-1\}$. Hence the factors $\prod_{i\in\overline{I(\varphi)}}(\varphi(t_i) - 1)^{-1}$ which appear in the three terms of (9.2.f) are identically the same. For a set $I \subset I(\varphi)$ not containing m, the corresponding summands in $\langle L, \varphi, k\rangle_2, \langle L', \varphi, k\rangle_2$ do not depend on the framing of the m-th component and therefore are equal. Consider the summand on the right-hand side of (9.2.a) corresponding to a set $I \subset I(\varphi)$ containing m. The matrices $\ell^I(L)$ and $\ell^I(L')$ coincide except at the (m, m)-entry where $\ell^I_{m,m}(L) = \ell^I_{m,m}(L') + 1$. It is easy to see that

$$\det(\ell^I(L)) - \det(\ell^I(L')) = \det(\ell^J(L^{\overline{m}}))$$

where $J = J_I = I\backslash\{m\} \subset I(\psi)$. The determinant $\det(\ell^J(L^{\overline{m}}))$ is exactly the determinant appearing in $\langle L^{\overline{m}}, \psi, k^{\overline{m}}\rangle_2$ in the summand corresponding to J. The factor $\varphi^{\overline{I}}_{\#}(\nabla(L^{\overline{I}}, L^{\overline{I}\cap I(\varphi)}, k^{\overline{I}}))$ is the same for both links L, L' and is equal to $\psi^{\tilde{J}}_{\#}(\nabla((L^{\overline{m}})^{\tilde{J}}, (L^{\overline{m}})^{\tilde{J}\cap I(\psi)}, (k^{\overline{m}})^{\tilde{J}}))$ where $\tilde{J} = \{1, ..., m-1\}\backslash J = \overline{I}$. Clearly, $(-1)^{|I|} = -(-1)^{|J|}$. Since the formula $I \mapsto J_I = I\backslash\{m\}$ establishes a bijective correspondence between sets $I \subset I(\varphi)$ containing m and subsets of $I(\psi)$, these equalities yield (9.2.f). Formulas (9.2.e), (9.2.f) and the induction assumption $\langle L^{\overline{m}}, \psi, k^{\overline{m}}\rangle_1 = \langle L^{\overline{m}}, \psi, k^{\overline{m}}\rangle_2$ imply (9.2.c). This accomplishes the proof of the theorem.

10. A surgery formula for the Alexander polynomial

10.1. The refined Alexander polynomial. Let M be a closed connected 3-manifold with $b_1(M) \geq 1$. Set $G = H_1(M)/\text{Tors}H_1(M)$. The Alexander polynomial $\Delta(M)$ of $\pi_1(M)$ is defined using a finite presentation of $\pi_1(M)$ and the Fox differential calculus (see [CF]). This polynomial is an element of the group ring $\mathbf{Z}[G]$ defined up to multiplication by $\pm G$. The theory of torsions allows to introduce a refinement $\Delta(M, e, \omega)$ of $\Delta(M)$ depending on an Euler structure $e \in \text{vect}(M)$ and a homology orientation ω of M. We define $\Delta(M, e, \omega)$ as follows. Denote by $Q(G)$ the field of fractions of the domain $\mathbf{Z}[G]$. Denote by μ the composition of the projection $\mathbf{Z}[H_1(M)] \to \mathbf{Z}[G]$ with the inclusion $\mathbf{Z}[G] \hookrightarrow Q(G)$. Set $\Delta(M, e, \omega) = \tau^\mu(M, e, \omega) \in Q(G)$. It follows from [Tu1] that $\Delta(M, e, \omega)$ is a refinement of $\Delta(M)$ in the following sense. If $b_1(M) \geq 2$ then $\Delta(M, e, \omega) \in \mathbf{Z}[G]$ and $\Delta(M) = \pm G\,\Delta(M, e, \omega)$. If $b_1(M) = 1$ and M is orientable then $\Delta(M, e, \omega) \in (t-1)^{-2}\mathbf{Z}[G]$ for any generator t of G and $\Delta(M) = \pm G(t-1)^2\Delta(M, e, \omega)$. A similar formula holds for non-orientable M (see [Tu1]) but we shall not need it.

10.2. Theorem (a surgery formula for $\Delta(M, e, \omega)$). *Let M be a closed 3-manifold with $b_1(M) \geq 1$ obtained by surgery on a framed oriented link $L =$*

$L_1 \cup ... \cup L_m$ in an oriented 3-dimensional integral homology sphere N. Let $k = (k_1, ..., k_m)$ be a charge on L. Let $[[t_1]], ..., [[t_m]] \in G = H_1(M)/\mathrm{Tors}H_1(M)$ be represented by the meridians of $L_1, ..., L_m$, respectively. Set

$$I_0 = \{i \mid [[t_i]] = 1\} \subset \{1, ..., m\}. \tag{10.2.a}$$

Then

$$\Delta(M, e_k^M, \omega_L^M) \tag{10.2.b}$$
$$= \prod_{i \in \overline{I_0}} ([[t_i]] - 1)^{-1} \sum_{I \subset I_0} (-1)^{|I|} \det(\ell^I(L))\, \mu_\#^{\overline{I}}(\nabla(L^{\overline{I}}, L^{\overline{I} \cap I_0}, k^{\overline{I}}))$$

where $\mu_\#^{\overline{I}} : (\mathbf{Z}[H(L^{\overline{I}}, L^{\overline{I} \cap I_0})])_1 \to Q(G)$ is the natural extension of the ring homomorphism $\mu^{\overline{I}} : \mathbf{Z}[H(L^{\overline{I}}, L^{\overline{I} \cap I_0})] \to \mathbf{Z}[G]$ sending the generator of $H(L^{\overline{I}}, L^{\overline{I} \cap I_0})$ represented by the meridian of L_i to $[[t_i]] \in G$, for all $i \in \overline{I}$.

Formula (10.2.b) is obtained by a direct application of Theorem 9.2 to the ring homomorphism μ defined in Sect. 10.1. Note that all $[[t_i]] - 1 \in \mathbf{Z}[G]$ with $i \in \overline{I_0}$ are non-zero and therefore invertible in $Q(G)$. We have

$$\mathrm{rank}\, H(L^{\overline{I}}, L^{\overline{I} \cap I_0}) \geq |\overline{I} \backslash (\overline{I} \cap I_0)| = m - |I_0| \geq b_1(M) \geq 1. \tag{10.2.c}$$

Hence $\nabla(L^{\overline{I}}, L^{\overline{I} \cap I_0}, k^{\overline{I}}) \in (\mathbf{Z}[H(L^{\overline{I}}, L^{\overline{I} \cap I_0})])_1$ and (10.2.b) makes sense.

If $m - |I_0| \geq 2$, then $\nabla(L^{\overline{I}}, L^{\overline{I} \cap I_0}, k^{\overline{I}}) \in \mathbf{Z}[H(L^{\overline{I}}, L^{\overline{I} \cap I_0})]$, and (10.2.b) can be rewritten in a slightly simpler form:

$$\Delta(M, e_k^M, \omega_L^M) \tag{10.2.d}$$
$$= \prod_{i \in \overline{I_0}} ([[t_i]] - 1)^{-1} \sum_{I \subset I_0} (-1)^{|I|} \det(\ell^I(L))\, \mu^{\overline{I}}(\nabla(L^{\overline{I}}, L^{\overline{I} \cap I_0}, k^{\overline{I}})).$$

This applies in particular if $b_1(M) \geq 2$.

If $m - |I_0| = 1$ then it may happen that $\mathrm{rank}\, H(L^{\overline{I}}, L^{\overline{I} \cap I_0}) = 1$. In this case we do need $\mu_\#^{\overline{I}}$. We shall see now that this can happen for only one entry, namely the one corresponding to $I = I_0$. We begin with a lemma which will be proven in Sect. 10.5.

10.3. Lemma. *Assume that I_0 consists of all $i = 1, ..., m$ except a certain n. Then the framing number of L_n is 0 and $lk(L_n, L_i) = 0$ for all $i \neq n$. For sets $I \subset J \subset I_0$, we have $\mathrm{rank}\, H(L^{\overline{I}}, L^{\overline{I} \cap J}) = 1$ if and only if $I = J = I_0$.*

Using this lemma we can reformulate (10.2.b) in the case $I_0 = \{1, ..., m\} \backslash \{n\}$. Set $t = [[t_n]] \in G$. By assumption, $t \neq 1$ and therefore $t - 1$ is invertible in $Q(G)$. It follows from Theorem 10.2, Lemma 10.3, and (8.1.c) that

$$\Delta(M, e_k^M, \omega_L^M) = (t-1)^{-1} \sum_{I \subset I_0, I \neq I_0} (-1)^{|I|} \det(\ell^I(L))\, \mu^{\overline{I}}(\nabla(L^{\overline{I}}, L^{\overline{I} \cap I_0}, k^{\overline{I}}))$$

$$+ (-1)^m \det(\ell^{I_0}(L))\; t^{(k_n+1)/2}\,(t-1)^{-2}\,\Delta_{L_n}(t) \qquad (10.2.e)$$

where Δ_{L_n} is the canonically normalized Alexander polynomial of L_n (see Sect. 8.1) and $\Delta_{L_n}(t) \in \mathbf{Z}[G]$ is obtained by computing Δ_{L_n} on t. By definition of a charge, the integer $k_n + 1 = k_n + 1 - lk(L_n, L\backslash L_n)$ is even.

10.4. Special cases. We describe two cases where (10.2.b) simplifies. If $m \geq 2$ and all $[[t_1]], ..., [[t_m]]$ are non-trivial elements of G then $I_0 = \emptyset$ and the sum on the right-hand side of (10.2.b) has only one term. We obtain

$$\Delta(M, e_k^M, \omega_L^M) = \prod_{i=1}^{m}([[t_i]] - 1)^{-1}\; [[\nabla(L, k)]]$$

where for a Laurent polynomial $\nabla \in \mathbf{Z}[t_1^{\pm 1}, ..., t_m^{\pm 1}]$ we denote by $[[\nabla]]$ the element of $\mathbf{Z}[G]$ obtained from ∇ by the substitution $t_i \mapsto [[t_i]] \in G$ for $i = 1, ..., m$.

Consider now the case where $L = L_1 \cup ... \cup L_m \subset N$ is an algebraically split link. This means that $lk(L_i, L_j) = 0$ for all $i \neq j$ where lk is the linking number in N. A charge $k = (k_1, ..., k_m)$ on L is just an m-tuple of odd integers. It follows easily from the Torres formula (5.1.a) and definitions that the Laurent polynomial $\nabla(L, k)$ is divisible by $\prod_{i=1}^m (t_i - 1)$ in $\mathbf{Z}[t_1^{\pm 1}, ..., t_m^{\pm 1}]$. Set

$$\check{\nabla}(L, k) = \nabla(L, k) / \prod_{i=1}^{m}(t_i - 1) \in \mathbf{Z}[t_1^{\pm 1}, ..., t_m^{\pm 1}].$$

Let M be a 3-manifold obtained by surgery along L endowed with certain framing numbers $f_1, ..., f_m \in \mathbf{Z}$. The set I_0 defined by (10.2.a) is computed by $I_0 = \{i = 1, ..., m \,|\, f_i \neq 0\}$. Clearly, $b_1(M) = m - |I_0|$. If $b_1(M) \geq 2$ then (10.2.d) implies

$$\Delta(M, e_k^M, \omega_L^M) = \sum_{I \subset I_0} (-1)^{|I|} \prod_{i \in I} f_i\; [[\check{\nabla}(L^{\overline{I}}, k^{\overline{I}})]].$$

If $b_1(M) = 1$ then I_0 consists of all $i = 1, ..., m$ except a certain n and (10.2.e) implies

$$\Delta(M, e_k^M, \omega_L^M) = \sum_{I \subset I_0, I \neq I_0} (-1)^{|I|} \prod_{i \in I} f_i\; [[\check{\nabla}(L^{\overline{I}}, k^{\overline{I}})]]$$

$$+ (-1)^m \big(\prod_{i \neq n} f_i\big)\; t^{(k_n+1)/2}\,(t-1)^{-2}\,\Delta_{L_n}(t)$$

where $t = [[t_n]] \in G$. In particular, if L is a knot with framing number 0 and charge $k \in \mathbf{Z}$ then $\Delta(M, e_k^M, \omega_L^M) = -t^{(k+1)/2}\,(t-1)^{-2}\,\Delta_L(t)$.

10.5. Proof of Lemma 10.3. By assumption, $[t_i] \in \mathrm{Tors} H_1(M)$ for all $i \neq n$ and $[t_n] \in H_1(M)$ is an element of infinite order. Therefore the group $H_1(M)/\{[t_i] = 1 \,|\, i \neq n\}$ is infinite. Since $(lk(L_i, L_j))_{i,j}$ is a presentation matrix

for $H_1(M)$ with respect to the generators $[t_1], ..., [t_m]$, this implies that $lk(L_n, L_i) = 0$ for all i. If $I = J = I_0$ then $H(L^{\overline{I}}, L^{\overline{I}\cap J}) = H(L_n, \emptyset) = \mathbf{Z}$. Conversely, assume that $I \subset J \subset I_0$ and $\operatorname{rank} H(L^{\overline{I}}, L^{\overline{I}\cap J}) = 1$. The group $H(L^{\overline{I}}, L^{\overline{I}\cap J})$ is generated by $\{t_i\}_{i\in\overline{I}}$ modulo the relations

$$\prod_{i\in\overline{I}} (t_i t_r^{-1})^{lk(L_i,L_r)} = 1$$

numerated by $r \in \overline{I}\cap J$. By the first part of the claim, these relations do not involve the generator t_n. Hence $H(L^{\overline{I}}, L^{\overline{I}\cap J})$ splits as a product of the infinite cyclic group generated by t_n and a group H' generated by t_i with $i \in \overline{I}\backslash\{n\} = I_0\backslash I$. If $I \neq I_0$ then there is an epimorphism $H' \to \mathbf{Z}$ sending all t_i with $i \in I_0\backslash I$ to 1. Then $\operatorname{rank} H(L^{\overline{I}}, L^{\overline{I}\cap J}) = \operatorname{rank} H' + 1 \geq 2$ which contradicts our assumptions. Thus, $I = I_0$ and therefore also $J = I_0$.

11. A surgery formula for $\tau(M)$ in the case $b_1(M) \geq 1$

11.1. Setting and notation. Let M be a closed 3-manifold obtained by surgery on a framed oriented link $L = L_1 \cup ... \cup L_m$ in an oriented 3-dimensional integral homology sphere N. Let $k = (k_1, ..., k_m)$ be a charge on L. Set $H = H_1(M)$. We shall need the following notation.

For any $a \in Q(H)$ we define the *reduced inverse* a_{red}^{-1} as follows. Recall that $Q(H)$ splits as a direct sum of fields, $Q(H) = \oplus_r F_r$. There is a unique expansion $a = \sum_r a_r$ with $a_r \in F_r$ for all r. Set $a_{\text{red}}^{-1} = \sum_{r, a_r\neq 0} a_r^{-1}$ where $a_r^{-1} \in F_r$ is the inverse of a_r in F_r. Clearly, if a is invertible in $Q(H)$ then $a_{\text{red}}^{-1} = a^{-1}$. The reduced inverse is defined for all elements of $Q(H)$. For instance, $0_{\text{red}}^{-1} = 0$. If t is an element of H of infinite order then $(t-1)_{\text{red}}^{-1} = (t-1)^{-1}$. If t is an element of H of finite order $n \geq 1$ then a simple computation shows that

$$(t-1)_{\text{red}}^{-1} = \frac{1 + 2t + 3t^2 + ... + nt^{n-1}}{n} - \frac{n+1}{2} \cdot \frac{1 + t + ... + t^{n-1}}{n}$$

$$= \frac{(1-n) + (3-n)t + (5-n)t^2 + ... + (n-1)t^{n-1}}{2n}.$$

Let $[t_1], ..., [t_m] \in H = H_1(M)$ be represented by the meridians of $L_1, ..., L_m$, respectively. Set

$$I_0 = \{i \,|\, [t_i] \in \operatorname{Tors} H\} \subset \{1, ..., m\}$$

(this is equivalent to (10.2.a)). For any sets $I \subset J \subset I_0$ we define an additive homomorphism from $\mathbf{Z}[H(L^{\overline{I}}, L^{\overline{I}\cap J})]$ to $\mathbf{Q}[H]$ called *transfer*. The image of $a \in \mathbf{Z}[H(L^{\overline{I}}, L^{\overline{I}\cap J})]$ under the transfer is denoted a^{tr}. By additivity, it suffices to define a^{tr} for $a \in H(L^{\overline{I}}, L^{\overline{I}\cap J})$. Consider the group $H_J = H/\{[t_j] = 1 \,|\, j \in J\}$. Let p be the projection $H \to H_J$. Sending each t_i with $i \in \overline{I}$ to $p([t_i]) \in H_J$, we obtain

a group homomorphism, $q : H(L^{\overline{I}}, L^{\overline{I}\cap J}) \to H_J$. For $a \in H(L^{\overline{I}}, L^{\overline{I}\cap J})$ we set $a^{\mathrm{tr}} = |\mathrm{Ker}\, p|^{-1} \sum_{h\in p^{-1}q(a)} h \in \mathbf{Q}[H]$ where $|\mathrm{Ker}\, p| = \mathrm{card}(\mathrm{Ker}\, p)$ and the addition on the right-hand side is the one in the group ring $\mathbf{Q}[H]$. The sum on the right-hand side is finite: the inclusion $J \subset I_0$ ensures that $\mathrm{Ker}\, p$ is finite. The transfer $\mathbf{Z}[H(L^{\overline{I}}, L^{\overline{I}\cap J})] \to \mathbf{Q}[H]$ is multiplicative but in general not a ring homomorphism since $1^{\mathrm{tr}} = |\mathrm{Ker}\, p|^{-1} \sum_{h\in \mathrm{Ker}\, p} h$.

We give now a surgery formula for $\tau(M, e_k^M, \omega_L^M)$ in the case $b_1(M) \geq 1$, see Appendix 1 for the case $b_1(M) = 0$.

11.2. Theorem. *If* $m - |I_0| \geq 2$, *then*

$$\tau(M, e_k^M, \omega_L^M) \tag{11.2.a}$$
$$= \sum_{I\subset J\subset I_0} (-1)^{|I|} \det(\ell^I(L)) \prod_{i\in \overline{J}} ([t_i]-1)^{-1}_{\mathrm{red}} \, (\nabla(L^{\overline{I}}, L^{\overline{I}\cap J}, k^{\overline{I}}))^{\mathrm{tr}}.$$

If I_0 *consists of all* $i = 1, ..., m$ *except a certain* n, *then*

$$\tau(M, e_k^M, \omega_L^M) \tag{11.2.b}$$
$$= \sum_{I\subset J\subset I_0, I\neq I_0} (-1)^{|I|} \det(\ell^I(L)) \prod_{i\in \overline{J}} ([t_i]-1)^{-1}_{\mathrm{red}} \, (\nabla(L^{\overline{I}}, L^{\overline{I}\cap J}, k^{\overline{I}}))^{\mathrm{tr}}$$
$$+(-1)^m \, \frac{\det(\ell^{I_0}(L))}{|\mathrm{Tors} H|} \Big(\sum_{h\in \mathrm{Tors} H} h\Big) [t_n]^{(k_n+1)/2} \, ([t_n]-1)^{-2} \, \Delta_{L_n}([t_n])$$

where Δ_{L_n} *is the canonically normalized Alexander polynomial of* L_n *(see Sect. 8.1) and* $\Delta_{L_n}([t_n]) \in \mathbf{Z}[H]$ *is obtained by computing* Δ_{L_n} *on* $[t_n]$.

Proof. Consider first the case $m - |I_0| \geq 2$. The sum on the right-hand side of (11.2.a) goes over all subsets I, J of I_0 such that $I \subset J$. By (10.2.c), we have $\mathrm{rank}\, H(L^{\overline{I}}, L^{\overline{I}\cap J}) \geq 2$ so that $\nabla(L^{\overline{I}}, L^{\overline{I}\cap J}, k^{\overline{I}}) \in \mathbf{Z}[H(L^{\overline{I}}, L^{\overline{I}\cap J})]$ and its transfer $(\nabla(L^{\overline{I}}, L^{\overline{I}\cap J}, k^{\overline{I}}))^{\mathrm{tr}}$ is a well defined element of $\mathbf{Q}[H] \subset Q(H)$.

Denote by φ_r the composition of the inclusion $\mathbf{Z}[H] \hookrightarrow Q(H)$ and the projection $Q(H) \to F_r$ on the r-th term in the splitting $Q(H) = \oplus_r F_r$. Set

$$I_r = I(\varphi_r) = \{ i \,|\, \varphi_r([t_i]) = 1\} \subset \{1, ..., m\}$$

and $\tau_r = \tau^{\varphi_r}(M, e_k^M, \omega_L^M) \in F_r$. Note that $I_r \subset I_0$ for all r and therefore $|\overline{I}_r| \geq |\overline{I}_0| \geq 2$. For $I \subset J \subset I_0$, set $\varphi_r^{\overline{I}} = (\varphi_r)^{\overline{I}} : \mathbf{Z}[H(L^{\overline{I}}, L^{\overline{I}\cap J})] \to F_r$ (cf. Sect. 9.1). We have

$$\tau(M, e_k^M, \omega_L^M) = \sum_r \tau_r$$
$$= \sum_r \sum_{I\subset I_r} \prod_{i\in \overline{I_r}} (\varphi_r([t_i])-1)^{-1} (-1)^{|I|} \det(\ell^I(L)) \, \varphi_r^{\overline{I}}(\nabla(L^{\overline{I}}, L^{\overline{I}\cap I_r}, k^{\overline{I}}))$$

$$= \sum_{I\subset J\subset I_0} (-1)^{|I|} \det(\ell^I(L)) \prod_{i\in \overline{J}}([t_i]-1)^{-1}_{\mathrm{red}} \sum_{r, J=I_r} \varphi_r^{\overline{I}}(\nabla(L^{\overline{I}}, L^{\overline{I}\cap J}, k^{\overline{I}}))$$

$$= \sum_{I\subset J\subset I_0} (-1)^{|I|} \det(\ell^I(L)) \prod_{i\in \overline{J}}([t_i]-1)^{-1}_{\mathrm{red}} \sum_{r, J\subset I_r} \varphi_r^{\overline{I}}(\nabla(L^{\overline{I}}, L^{\overline{I}\cap J}, k^{\overline{I}})).$$

Here the first equality is the definition of $\tau(M, e_k^M, \omega_L^M)$. The second equality follows from Theorem 9.2. The third equality is a tautology (the projections of both sides to each F_r are equal). The fourth equality follows from the fact that the terms on the right-hand side corresponding to proper subsets $J\subset I_r$ are equal to 0. Indeed for $i\in I_r\backslash J$, the projection of $[t_i]-1$ and hence of $([t_i]-1)^{-1}_{\mathrm{red}}$ to F_r is equal to 0. Therefore $([t_i]-1)^{-1}_{\mathrm{red}}\,\mathrm{Im}(\varphi_r^{\overline{I}}) = 0$. The theorem now follows from the next claim.

Claim. For any sets $I\subset J\subset I_0$ and any $a\in \mathbf{Z}[H(L^{\overline{I}}, L^{\overline{I}\cap J})]$,

$$\sum_{r, J\subset I_r} \varphi_r^{\overline{I}}(a) = a^{\mathrm{tr}}.$$

Proof. By additivity, it suffices to consider the case where $a\in H(L^{\overline{I}}, L^{\overline{I}\cap J})$. Let $H = H_1(M)$ and H_J, p, q be the same objects as in Sect. 11.1. Since $\mathrm{Ker}\, p$ is a subgroup of $\mathrm{Tors}\, H$, the epimorphism p extends to a ring epimorphism $\tilde{p} : Q(H)\to Q(H_J)$. Moreover, the splitting of $Q(H_J)$ into a direct sum of fields is obtained from the splitting of $Q(H)$ by quotienting out all F_r such that $\varphi_r(\mathrm{Ker}\, p)\neq 1$. Note that $\varphi_r(\mathrm{Ker}\, p) = 1$ if and only if $J\subset I_r$. Thus

$$Q(H_J) = \bigoplus_{r, \varphi_r(\mathrm{Ker}\, p)=1} F_r = \bigoplus_{r, J\subset I_r} F_r.$$

For each r such that $J\subset I_r$, the composition of $\tilde{p} : Q(H)\to Q(H_J)$ with the projection $\psi_r : Q(H_J)\to F_r$ is equal to the projection $Q(H)\to F_r$.

The identity $\sum_r \varphi_r(h) = h$ for all $h\in H$ implies that

$$a^{\mathrm{tr}} = |\mathrm{Ker}\, p|^{-1} \sum_{h\in p^{-1}q(a)} h = |\mathrm{Ker}\, p|^{-1} \sum_r \varphi_r(\sum_{h\in p^{-1}q(a)} h).$$

Observe that if $\varphi_r(\mathrm{Ker}\, p)\neq 1$ then $\varphi_r(\sum_{h\in p^{-1}q(a)} h) = 0$. Therefore

$$a^{\mathrm{tr}} = |\mathrm{Ker}\, p|^{-1} \sum_{r, \varphi_r(\mathrm{Ker}\, p)=1} \varphi_r(\sum_{h\in p^{-1}q(a)} h)$$

$$= |\mathrm{Ker}\, p|^{-1} \sum_{r, J\subset I_r} \psi_r\tilde{p}(\sum_{h\in p^{-1}q(a)} h) = \sum_{r, J\subset I_r} \psi_r(q(a)) = \sum_{r, J\subset I_r} \varphi_r^{\overline{I}}(a)$$

where the last formula follows from the definition of $\varphi_r^{\overline{I}} = (\varphi_r)^{\overline{I}}$.

The proof of (11.2.b) is similar with a separate analysis of the term corresponding to $I = J = I_0$ using Lemma 10.3.

11.3. Remarks. 1. It follows from Lemma 10.3 that $\det(\ell^{I_0}(L))/|\mathrm{Tors} H| = \pm 1$ in (11.2.b). This sign will also appear in Appendix 2.

2. As an exercise, the reader may deduce Theorem 10.2 from Theorem 11.2. Hint: use the fact that $\Delta(M, e_k^M, \omega_L^M)$ is the image of $\tau(M, e_k^M, \omega_L^M)$ under the projection $Q(H) \to Q(H/\mathrm{Tors}\, H)$ and that all summands corresponding to $J \neq I_0$ are annihilated by this projection.

11.4. Special cases. We discuss two special cases of Theorem 11.2. We use the notation of Sect. 11.1. For a Laurent polynomial $\nabla \in \mathbf{Z}[t_1^{\pm 1}, ..., t_m^{\pm 1}]$, we denote by $[\nabla]$ the element of $\mathbf{Z}[H]$ obtained from ∇ by the substitution $t_i \mapsto [t_i] \in H$ where $i = 1, ..., m$. Here is the simplest case of (11.2.a): if all $[t_1], ..., [t_m]$ have infinite order in H then $I_0 = \emptyset$ and (11.2.a) yields

$$\tau(M, e_k^M, \omega_L^M) = \prod_{i=1}^{m} ([t_i] - 1)^{-1} \, [\nabla(L, k)]. \tag{11.4.a}$$

Consider now the case where L is algebraically split. Let $f_i \in \mathbf{Z}$ be the framing number of L_i for $i = 1, ..., m$. For $i \in I_0 = \{i = 1, ..., m \,|\, f_i \neq 0\}$, the element $[t_i] \in H$ has order $|f_i| = \mathrm{sign}(f_i) f_i$ where $\mathrm{sign}(f_i) = \pm 1$ is the sign of f_i. Set

$$s_i = \mathrm{sign}(f_i)(1 + [t_i] + [t_i]^2 + ... + [t_i]^{|f_i| - 1}) \in \mathbf{Z}[H].$$

Assume first that $b_1(M) = m - |I_0| \geq 2$. As we show below, formula (11.2.a) can be rewritten in this case as follows:

$$\tau(M, e_k^M, \omega_L^M) = \sum_{I \subset I_0} (-1)^{|I|} \, (\prod_{i \in I} s_i) \, [\check{\nabla}(L^{\overline{I}}, k^{\overline{I}})]. \tag{11.4.b}$$

Here $k^{\overline{I}}$ is just the restriction of k to $L^{\overline{I}} = \cup_{i \in \overline{I}} L_i$. The assumption $b_1(M) \geq 2$ ensures that $L^{\overline{I}}$ has ≥ 2 components for any $I \subset I_0$ so that $\check{\nabla}(L^{\overline{I}}, k^{\overline{I}})$ is a Laurent polynomial on $\{t_i\}_{i \in \overline{I}}$ and $[\check{\nabla}(L^{\overline{I}}, k^{\overline{I}})]$ is a well defined element of $\mathbf{Z}[H]$. The sum on the right-hand side of (11.4.b) contains a term corresponding to $I = \emptyset$ and equal to $[\check{\nabla}(L, k)]$. The other terms correspond to proper sublinks of L. For instance, if $f_i = 0$ for all i then $I_0 = \emptyset$ and we obtain a formula $\tau(M, e_k^M, \omega_L^M) = [\check{\nabla}(L, k)]$ equivalent to (11.4.a). If I_0 has only one element i then (11.4.b) gives

$$\tau(M, e_k^M, \omega_L^M) = [\check{\nabla}(L, k)] - s_i \, [\check{\nabla}(L^{\overline{i}}, k^{\overline{i}})]$$

where $\overline{i} = \{1, ..., m\} \backslash i$. If I_0 has two elements i, j then (11.4.b) gives

$$\tau(M, e_k^M, \omega_L^M) = [\check{\nabla}(L, k)] - s_i \, [\check{\nabla}(L^{\overline{i}}, k^{\overline{i}})] - s_j \, [\check{\nabla}(L^{\overline{j}}, k^{\overline{j}})] + s_i s_j \, [\check{\nabla}(L^{\overline{\{i,j\}}}, k^{\overline{\{i,j\}}})].$$

Let us deduce (11.4.b) from (11.2.a). Observe that for $I \subset J \subset I_0$,

$$\det(\ell^I(L)) = \prod_{i \in I} f_i.$$

It follows from definitions that $\nabla(L^{\overline{I}}, L^{\overline{I} \cap J}, k^{\overline{I}}) = \prod_{i \in \overline{J}} (t_i - 1)\ \check{\nabla}(L^{\overline{I}}, k^{\overline{I}})$. Hence

$$(\nabla(L^{\overline{I}}, L^{\overline{I} \cap J}, k^{\overline{I}}))^{\mathrm{tr}} = \prod_{j \in J} \sigma_j \prod_{i \in \overline{J}} ([t_i] - 1)\ [\check{\nabla}(L^{\overline{I}}, k^{\overline{I}})]$$

where

$$\sigma_j = |f_j|^{-1}(1 + [t_j] + ... + [t_j]^{|f_j|-1}) = f_j^{-1} s_j. \tag{11.4.c}$$

Formula (11.2.a) yields then

$$\tau(M, e_k^M, \omega_L^M) \tag{11.4.d}$$

$$= \sum_{I \subset J \subset I_0} (-1)^{|I|} \prod_{i \in I} f_i \prod_{i \in \overline{J}} ([t_i] - 1)^{-1}_{\mathrm{red}} \prod_{j \in J} \sigma_j \prod_{i \in \overline{J}} ([t_i] - 1)\ [\check{\nabla}(L^{\overline{I}}, k^{\overline{I}})].$$

Observe that for $i \in \overline{I}_0$, we have $([t_i] - 1)^{-1}_{\mathrm{red}}([t_i] - 1) = ([t_i] - 1)^{-1}([t_i] - 1) = 1$. If $i \in I_0$ then $([t_i] - 1)^{-1}_{\mathrm{red}}([t_i] - 1) = 1 - \sigma_i$: both sides are mapped to 0 by any character of Tors$H$ mapping t_i to 1 and are mapped to 1 by all other characters of Tors$H$. Therefore the right-hand side of (11.4.d) can be rewritten as

$$\sum_{I \subset J \subset I_0} (-1)^{|I|} \prod_{i \in I} f_i \prod_{j \in J} \sigma_j \prod_{i \in I_0 \setminus J} (1 - \sigma_i)\, [\check{\nabla}(L^{\overline{I}}, k^{\overline{I}})]$$

$$= \sum_{I \subset I_0} (-1)^{|I|} \prod_{i \in I} f_i \Big(\sum_{J, I \subset J \subset I_0} \prod_{j \in J} \sigma_j \prod_{i \in I_0 \setminus J} (1 - \sigma_i) \Big)\, [\check{\nabla}(L^{\overline{I}}, k^{\overline{I}})].$$

The sum over J obviously equals to $\prod_{i \in I} \sigma_i$. Therefore

$$\tau(M, e_k^M, \omega_L^M) = \sum_{I \subset I_0} (-1)^{|I|} \prod_{i \in I} f_i \prod_{i \in I} \sigma_i\, [\check{\nabla}(L^{\overline{I}}, k^{\overline{I}})].$$

Substituting $\sigma_i = f_i^{-1} s_i$ we obtain (11.4.b).

Assume now that L is algebraically split and $b_1(M) = m - |I_0| = 1$. Let n be the only element of $\overline{I_0}$. Then (11.2.b) can be similarly rewritten as follows:

$$\tau(M, e_k^M, \omega_L^M) = \sum_{I \subset I_0, I \neq I_0} (-1)^{|I|} \prod_{i \in I} s_i\ [\check{\nabla}(L^{\overline{I}}, k^{\overline{I}})] \tag{11.4.e}$$

$$+(-1)^m \Big(\prod_{i \in I_0} s_i \Big)\, [t_n]^{(k_n+1)/2}\, ([t_n] - 1)^{-2}\, \Delta_{L_n}([t_n]).$$

12. A surgery formula for the Seiberg-Witten invariants of 3-manifolds

12.1. Torsion versus SW-invariants. Let M be a closed, connected, oriented 3-manifold with homology orientation ω. Set $H = H_1(M)$. Assume first that $b_1(M) \geq 2$. The Seiberg-Witten invariant of M is a $\mathbf{Z}$-valued function $e \mapsto SW(e, \omega)$ on $\mathrm{vect}(M)$, see for instance [MT]. This function has a finite support. Note that usually one considers the SW-invariants of $Spin^c$-structures on M; by [Tu5], the set of $Spin^c$-structures on M can be identified with $\mathrm{vect}(M) = \mathrm{Eul}(M)$. The SW-invariant is computed from the torsion τ as follows. Every element $a \in \mathbf{Z}[H]$ expands uniquely as a finite sum $\sum_{h\in H} a_h h$ with $a_h \in \mathbf{Z}$. By [Tu6] (see also Appenidx 3), $\tau(M, e, \omega) \in \mathbf{Z}[H]$ for any $e \in \mathrm{vect}(M)$ and

$$SW(e, \omega) = \pm(\tau(M, e, \omega))_1 \in \mathbf{Z} \tag{12.1.a}$$

where 1 is the neutral element of H and the sign $\pm$ does not depend on e. Together with (11.2.a) this implies the following formula for $\pm SW(e^M_k, \omega^M_L)$ in the setting of Sect. 11.1:

$$\pm SW(e^M_k, \omega^M_L) = \sum_{I\subset J\subset I_0} (-1)^{|I|} \det(\ell^I(L)) \left(\prod_{i\in\overline{J}} ([t_i]-1)^{-1}_{\mathrm{red}} \, (\nabla(L^{\overline{I}}, L^{\overline{I}\cap J}, k^{\overline{I}}))^{\mathrm{tr}} \right)_1$$

where the sum runs over all subsets I, J of I_0 such that $I \subset J$. This formula computes $\pm SW(e^M_k, \omega^M_L)$ in terms of the Conway polynomials of L and its sublinks, and the linking and framing numbers of the components of L.

Assume now that $b_1(M) = 1$. Choose an element $t \in H$ whose projection to $H/\mathrm{Tors}\, H = \mathbf{Z}$ is a generator. An element $g \in H$ is said to be t-positive (resp. t-negative) if $g \in t^K \mathrm{Tors}\, H$ with $K > 0$ (resp. with $K < 0$). The Seiberg-Witten invariant of M is a $\mathbf{Z}$-valued function $e \mapsto SW(e, \omega, t)$ on $\mathrm{vect}(M)$ depending on $t (\mathrm{mod}\, \mathrm{Tors}\, H) \in H/\mathrm{Tors}\, H$. It is computed from $\tau(M, e, \omega)$ as follows. Let Q be the subring of $Q(H)$ generated by $\mathbf{Q}[H]$ and elements $(g-1)^{-1}$ where g runs over elements of H of infinite order. Using the formula $(g-1)^{-1} = -1 - g - g^2 - ...$ for t-positive g and the formula $(g-1)^{-1} = -g^{-1}(g^{-1}-1)^{-1} = g^{-1} + g^{-2} + ...$ for t-negative g we can uniquely expand any $a \in Q$ as a formal series $a^t = \sum_{h\in H} a^t_h h$ with $a^t_h \in \mathbf{Q}$. The support of the map $h \mapsto a^t_h : H \to \mathbf{Q}$ is essentially t-positive in the sense that it meets the set of t-negative elements of H in a finite set. Theorem 11.2 or the results of [Tu5, Sect. 4.2] show that $\tau(M, e, \omega) \in Q$. It follows from [Tu6] (cf. Appendix 3 of the present paper) that

$$\pm SW(e, \omega, t) = (\tau(M, e, \omega))^{t^{-1}}_1 = (\tau(M, e^{-1}, \omega))^t_1. \tag{12.1.b}$$

Recall that $(e^M_k)^{-1} = e^M_{2-k}$, cf. Sect. 1.7. Combining this with Theorem 11.2 we obtain the following formulas for $\pm SW(e^M_k, \omega^M_L, t)$ in the setting of Sect. 11.1. If $m - |I_0| \geq 2$ (and $b_1(M) = 1$), then

$$\pm SW(e^M_k, \omega^M_L, t)$$

$$= \sum_{I\subset J\subset I_0} (-1)^{|I|} \det(\ell^I(L)) \left(\prod_{i\in \overline{J}} ([t_i]-1)^{-1}_{\rm red} \, (\nabla(L^{\overline{I}}, L^{\overline{I}\cap J}, k^{\overline{I}}))^{\rm tr} \right)_1^{t^{-1}}$$

$$= \sum_{I\subset J\subset I_0} (-1)^{|I|} \det(\ell^I(L)) \left(\prod_{i\in \overline{J}} ([t_i]-1)^{-1}_{\rm red} \, (\nabla(L^{\overline{I}}, L^{\overline{I}\cap J}, (2-k)^{\overline{I}}))^{\rm tr} \right)_1^{t}.$$

Note that $(2-k)^{\overline{I}} = 2 + (-k^{\overline{I}})$. If I_0 consists of all $i = 1, ..., m$ except a certain n, then $t = [t_n] \in H$ projects to a generator of $H/\mathrm{Tors}\, H$. Let $\Delta_{L_n}(t) = \sum_l z_l t^l$ be the canonically normalized Alexander polynomial of L_n. Then

$$\pm SW(e^M_k, \omega^M_L, t)$$

$$= \sum_{I\subset J\subset I_0, I\neq I_0} (-1)^{|I|} \det(\ell^I(L)) \left(\prod_{i\in \overline{J}} ([t_i]-1)^{-1}_{\rm red} \, (\nabla(L^{\overline{I}}, L^{\overline{I}\cap J}, k^{\overline{I}}))^{\rm tr} \right)_1^{t^{-1}}$$

$$+(-1)^m \frac{\det(\ell^{I_0}(L))}{|\mathrm{Tors} H|} (z_{(k_n-3)/2} + 2z_{(k_n-5)/2} + 3z_{(k_n-7)/2} + ...)$$

$$= \sum_{I\subset J\subset I_0, I\neq I_0} (-1)^{|I|} \det(\ell^I(L)) \left(\prod_{i\in \overline{J}} ([t_i]-1)^{-1}_{\rm red} \, (\nabla(L^{\overline{I}}, L^{\overline{I}\cap J}, (2-k)^{\overline{I}}))^{\rm tr} \right)_1^{t}$$

$$+(-1)^m \frac{\det(\ell^{I_0}(L))}{|\mathrm{Tors} H|} (z_{(k_n-3)/2} + 2z_{(k_n-5)/2} + 3z_{(k_n-7)/2} + ...).$$

12.2. The case of algebraically split links. The surgery formula for $SW(e^M_k, \omega^M_L)$ can be made quite explicit when the link L is algebraically split. Observe first that for an algebraically split link with $m \geq 2$ components $L = L_1 \cup ... \cup L_m$ in a homology sphere, the Laurent polynomial ∇_L is divisible by $\prod_{i=1}^m (t_i^2 - 1)$ in $\mathbf{Z}[t_1^{\pm 1}, ..., t_m^{\pm 1}]$. Thus we have a finite expansion

$$\nabla_L(t_1, ..., t_m) / \prod_{i=1}^m (t_i^2 - 1) = \sum_{l=(l_1,...,l_m)\in \mathbf{Z}^m} z_l(L) \; t_1^{l_1} ... t_m^{l_m}$$

where $z_l(L) \in \mathbf{Z}$.

Assume that a 3-manifold M is obtained by surgery on a framed oriented algebraically split link $L = L_1 \cup ... \cup L_m$ in an oriented 3-dimensional integral homology sphere. Let $k = (k_1, ..., k_m)$ be a charge on L. Let $f = (f_1, ..., f_m)$ be the tuple of the framing numbers of $L_1, ..., L_m$. Denote by J_0 the subset of $\{1, ..., m\}$ consisting of all j such that $f_j = 0$. Note that $|J_0| = b_1(M)$. As usual we distinguish two cases $b_1(M) \geq 2$ and $b_1(M) = 1$.

12.2.1. Case $b_1(M) \geq 2$. We claim that

$$\pm SW(e_k^M, \omega_L^M) \tag{12.2.a}$$
$$= \sum_{J_0 \subset J \subset \{1,...,m\}} (-1)^{|J|} \prod_{i \in \overline{J}} \mathrm{sign}(f_i) \sum_{l \in \mathbf{Z}^J, l \equiv -k (\mathrm{mod}\, 2f)} z_l(L^J).$$

To see this, we analyze the terms of (11.4.b) giving rise to entries of the neutral element in $\tau(M, e_k^M, \omega_L^M)$. By definition (see Sect. 8.1 and 10.4),

$$\check{\nabla}(L,k) = \nabla(L,k)/\prod_{i=1}^m (t_i - 1) = -t_1^{k_1/2} ... t_m^{k_m/2}\, \nabla_L(t_1^{1/2}, ..., t_m^{1/2})/\prod_{i=1}^m (t_i - 1)$$
$$= -\sum_{l=(l_1,...,l_m) \in \mathbf{Z}^m} z_l(L)\; t_1^{(k_1+l_1)/2} ... t_m^{(k_m+l_m)/2} \in \mathbf{Z}[t_1^{\pm 1}, ..., t_m^{\pm 1}].$$

Similar formulas hold for $\check{\nabla}(L^{\overline{I}}, k^{\overline{I}})$ where I runs over subsets of $I_0 = \overline{J_0}$. We substitute these formulas into (11.4.b) and use the obvious fact that a monomial $t_1^{n_1} ... t_m^{n_m}$ represents $1 \in H_1(M)$ iff $n_i \in \mathbf{Z}$ is divisible by f_i for all $i = 1, ..., m$. This implies that $1 \in H_1(M)$ appears in $\tau(M, e_k^M, \omega_L^M)$ with coefficient

$$-\sum_{I \subset I_0} (-1)^{|I|} \prod_{i \in I} \mathrm{sign}(f_i) \sum_{l \in \mathbf{Z}^{\overline{I}}, l \equiv -k (\mathrm{mod}\, 2f)} z_l(L^{\overline{I}}).$$

Setting here $I = \overline{J}$ we obtain

$$(-1)^{m-1} \sum_{J_0 \subset J \subset \{1,...,m\}} (-1)^{|J|} \prod_{i \in \overline{J}} \mathrm{sign}(f_i) \sum_{l \in \mathbf{Z}^J, l \equiv -k (\mathrm{mod}\, 2f)} z_l(L^J).$$

This and the results of [Tu6] mentioned in Sect. 12.1 imply (12.2.a).

The right-hand side of (12.2.a) contains two distinguished terms corresponding to $J = \{1, ..., m\}$ and to $J = J_0$. The term corresponding to $J = \{1, ..., m\}$ equals $(-1)^m \sum_{l \in \mathbf{Z}^m, l \equiv -k (\mathrm{mod}\, 2f)} z_l(L)$. The term corresponding to $J = J_0$ equals

$$(-1)^{b_1(M)} \Big(\prod_{j \in \overline{J}_0} \mathrm{sign}(f_j)\Big)\, z_{-k^{J_0}}(L^{J_0})$$

where $-k^J = \{-k_j\}_{j \in J}$ for any set $J \subset \{1, ..., m\}$. For example, if all f_j are equal to zero, then $J_0 = \{1, ..., m\}$ and

$$SW(e_k^M, \omega_L^M) = \pm\, z_{-k}(L). \tag{12.2.b}$$

If J_0 consists of all indices $1, ..., m$ except a certain i, i.e., $J_0 = \overline{i} = \{1, ..., m\} \backslash i$ then (12.2.a) gives

$$\pm\, SW(e_k^M, \omega_L^M) = \sum_{l \in \mathbf{Z}^m, l \equiv -k (\mathrm{mod}\, 2f)} z_l(L) - \mathrm{sign}(f_i)\, z_{-k^{\overline{i}}}(L^{\overline{i}}). \tag{12.2.c}$$

The condition $l \equiv -k (\operatorname{mod} 2f)$ means here that $l_j = -k_j$ for $j \neq i$ and $l_i \equiv -k_i (\operatorname{mod} 2f_i)$. If $J_0 = \overline{\{i,j\}}$ is the complement of two indices i, j then (12.2.a) gives

$$\pm\, SW(e^M_k, \omega^M_L) = \sum_{l \in \mathbf{Z}^m, l \equiv -k (\operatorname{mod} 2f)} z_l(L) - \operatorname{sign}(f_i) \sum_{l \in \mathbf{Z}^{\overline{i}}, l \equiv -k (\operatorname{mod} 2f)} z_l(L^{\overline{i}})$$

$$- \operatorname{sign}(f_j) \sum_{l \in \mathbf{Z}^{\overline{j}}, l \equiv -k (\operatorname{mod} 2f)} z_l(L^{\overline{j}}) + \operatorname{sign}(f_i) \operatorname{sign}(f_j)\, z_{-k^{\overline{\{i,j\}}}}(L^{\overline{\{i,j\}}}).$$

12.2.2. Case $b_1(M) = 1$. Let n be the only element of J_0. Let $\Delta_{L_n}(t) = \sum_l z_l t^l$ be the normalized Alexander polynomial of L_n. Then it follows similarly from (11.4.e) that

$$\pm SW(e^M_k, \omega^M_L, [t_n])$$

$$= \sum_{n \in J \subset \{1,\dots,m\}, |J| \geq 2} (-1)^{|J|} \prod_{i \in \overline{J}} \operatorname{sign}(f_i) \sum_{l \in \mathbf{Z}^J, l \equiv 2-k (\operatorname{mod} 2f)} z_l(L^J)$$

$$- \prod_{i \neq n} \operatorname{sign}(f_i)\, (z_{(k_n-3)/2} + 2z_{(k_n-5)/2} + 3z_{(k_n-7)/2} + \dots).$$

12.3. Examples. 1. Let a 3-manifold M be obtained by surgery along an oriented knot $L = L_1 \subset S^3$ with framing 0. Then for any odd integer k,

$$SW(e^M_k, \omega^M_L, [t_1]) = \pm (z_{(k_n-3)/2} + 2z_{(k_n-5)/2} + 3z_{(k_n-7)/2} + \dots)$$

where $\Delta_{L_n}(t) = \sum_l z_l t^l$.

2. Consider the Borromean link $L = L_1 \cup L_2 \cup L_3$. It is algebraically split, $z_l(L) = 0$ for all $l \neq 0$ and $z_0(L) = \pm 1$ where the sign depends on the orientation of L. All 2-component sublinks of L are trivial and have a zero Alexander-Conway polynomial. Therefore if M is obtained by surgery on L with framing numbers $(f \in \mathbf{Z}, 0, 0)$ then formulas (12.2.b), (12.2.c) show that there is an Euler structure on M with SW-invariant ± 1 and all other Euler structures on M have a zero SW-invariant. This was previously known for $f = 0$ where $M = S^1 \times S^1 \times S^1$.

Appendix 1. A surgery formula for rational homology spheres

We discuss here an analogue of Theorem 11.2 in the case where M is a rational homology sphere, i.e., $b_1(M) = 0$. We use the notation of Sect. 11.1. It is clear that $I_0 = \{1, \dots, m\}$ and $Q(H) = \mathbf{Q}[H]$ where $H = H_1(M)$. For every $i = 1, \dots, m$, we define $\sigma_i \in \mathbf{Q}[H]$ by (11.4.c) where $f_i \geq 1$ is the order of $[t_i] \in H$. For sets $I \subset J \subset \{1, \dots, m\}$, set $rk(I, J) = \operatorname{rank} H(L^{\overline{I}}, L^{\overline{I} \cap J})$.

The arguments of Sect. 11 apply here with the following changes. One of the summands in the splitting of $\mathbf{Q}[H]$ into a direct sum of fields corresponds to the augmentation homomorphism $\varphi_0 : \mathbf{Q}[H] \to \mathbf{Q}$ which maps H to 1. Clearly, $H_*^{\varphi_0}(M) = H_*(M;\mathbf{Q}) \neq 0$ so that $\tau^{\varphi_0}(M) = 0$. This allows us to proceed as in the proof of Theorem 11.2 involving only r such that $\varphi_r(H) \neq 1$. This leads to the condition $J \neq \{1,...,m\}$ or equivalently $|J| \leq m-1$ in the formulas below.

The second subtlety comes from the fact that the expression $(\nabla(L^{\overline{I}}, L^{\overline{I}\cap J}, k^{\overline{I}}))^{\rm tr}$ is defined only if $rk(I,J) \geq 2$. Note that $rk(I,J) \geq |\overline{J}| \geq 1$. We can have $rk(I,J) = 1$ only when $J = \overline{n} = \{1,...,m\}\backslash\{n\}$ for a certain $n = 1,...,m$. For a set $I \subset J = \overline{n}$ three cases may occur: (i) $rk(I,J) \geq 2$; (ii) $rk(I,J) = 1$ and $I \neq J$, and (iii) $I = J$. In the second case

$$([t_n]-1)\nabla(L^{\overline{I}}, L^{\overline{I}\cap J}, k^{\overline{I}}) \in \mathbf{Z}[H(L^{\overline{I}}, L^{\overline{I}\cap J})]$$

and we can set

$$(\nabla(L^{\overline{I}}, L^{\overline{I}\cap J}, k^{\overline{I}}))^{\rm tr} = ([t_n]-1)^{-1}_{\rm red}\,(([t_n]-1)\nabla(L^{\overline{I}}, L^{\overline{I}\cap J}, k^{\overline{I}}))^{\rm tr} \in \mathbf{Q}[H].$$

Then the same arguments as in the proof of Theorem 11.2 show that

$$\tau(M, e_k^M, \omega_L^M) =$$

$$\sum_{I\subset J\subset\{1,...,m\}, |I|\leq m-2, |J|\leq m-1} (-1)^{|I|} \det(\ell^I(L)) \prod_{i\in\overline{J}} ([t_i]-1)^{-1}_{\rm red}\,(\nabla(L^{\overline{I}}, L^{\overline{I}\cap J}, k^{\overline{I}}))^{\rm tr}$$

$$+(-1)^m \sum_{n=1}^m \det(\ell^{\overline{n}}(L))\,(\prod_{i\neq n}\sigma_i)\,[t_n]^{(k_n - lk(L_n, L^{\overline{n}})+1)/2}\,(([t_n]-1)^{-1}_{\rm red})^2\,\Delta_{L_n}([t_n]).$$

In the case where L is algebraically split and its components have non-zero framings $f_1,...,f_m$, we obtain

$$\tau(M, e_k^M, \omega_L^M) = \sum_{I\subset\{1,...,m\}, |I|\leq m-2} (-1)^{|I|} \prod_{i\in I} s_i\ [\check{\nabla}(L^{\overline{I}}, k^{\overline{I}})]$$

$$+(-1)^m \sum_{n=1}^m \big(\prod_{i\neq n} f_i\sigma_i\big)\,[t_n]^{(k_n+1)/2}\,(([t_n]-1)^{-1}_{\rm red})^2\,\Delta_{L_n}([t_n]).$$

Appendix 2. Computation of ω_L^M

Canonical homology orientation. Every closed oriented 3-manifold M has a canonical homology orientation ω_M determined by an arbitrary basis in the vector space $H_0(M;\mathbf{R}) \oplus H_1(M;\mathbf{R})$ followed by the Poincaré dual basis in $H_2(M;\mathbf{R}) \oplus H_3(M;\mathbf{R})$. Clearly, $\omega_{-M} = (-1)^{b_0(M)+b_1(M)}\omega_M$. The aim of this appendix is

to compute the homology orientation $\omega_L^M = (\omega_L)^M$ of M which appears above via ω_M. It is obvious that $\omega_L^M = \pm\omega_M$ and we shall compute the sign $\pm$ in this formula. We begin with a purely algebraic definition.

Sign of the determinant as a torsion. Let B be a symmetric $(m \times m)$-matrix over $\mathbf{R}$ with $m = 1, 2, ...$ Let b be the symmetric bilinear form on $\mathbf{R}^m$ determined by B. Quotienting $\mathbf{R}^m$ by the annihilator $\mathrm{Ann}(b)$ we obtain a non-degenerate symmetric bilinear form on $\mathbf{R}^m/\mathrm{Ann}(b)$. It can be represented with respect to a basis of $\mathbf{R}^m/\mathrm{Ann}(b)$ by a non-degenerate symmetric matrix. The sign of its determinant does not depend on the choice of the basis. This sign is denoted by $\det_0(B)$. If B is non-degenerate then $\det_0(B) = \mathrm{sign}(\det(B))$.

We can interpret $\det_0(B)$ in terms of torsions as follows. Consider the sequence of vector spaces and linear homomorphisms

$$C = (\mathrm{Ann}(b) \hookrightarrow \mathbf{R}^m \xrightarrow{\mathrm{ad}(b)} (\mathbf{R}^m)^* \to (\mathrm{Ann}(b))^*)$$

where $\mathrm{ad}(b) : \mathbf{R}^m \to (\mathbf{R}^m)^* = \mathrm{Hom}(\mathbf{R}^m, \mathbf{R})$ is adjoint to b and the homomorphism $(\mathbf{R}^m)^* \to (\mathrm{Ann}(b))^*$ is obtained by restricting linear functionals on $\mathbf{R}^m$ to $\mathrm{Ann}(b)$. Clearly, C is an acyclic chain complex. We provide $\mathrm{Ann}(b), \mathbf{R}^m$ with arbitrary bases and provide $(\mathrm{Ann}(b))^*, (\mathbf{R}^m)^*$ with dual bases. It follows from definitions that the sign $\tau_0(C) = \pm 1$ of the corresponding torsion $\tau(C) \in \mathbf{R}$ depends only on b and does not depend on the choice of bases in $\mathrm{Ann}(b), \mathbf{R}^m$. Choosing these bases so that the one in $\mathbf{R}^m$ extends the one in $\mathrm{Ann}(b)$ we easily compute that

$$\tau_0(C) = (-1)^{(m+1)\dim(\mathrm{Ann}(b))} \det_0(B). \tag{2.a}$$

Lemma. *Let M be a closed 3-manifold obtained by surgery on a framed oriented link $L = L_1 \cup ... \cup L_m$ in an oriented 3-dimensional integral homology sphere N. Let ω_M be the canonical homology orientation of M determined by the orientation of M induced by the one in N. Let B be the symmetric linking matrix $[lk(L_i, L_j)]_{i,j=1}^m$ where $lk(L_i, L_i)$ is the framing number of L_i. Then*

$$\omega_L^M = (-1)^{b_1(M)+m+1} \det_0(B)\, \omega_M. \tag{2.b}$$

Proof. We shall use the symbols E, U_i, Z_i introduced in Sect. 1.7. Let D_i be the meridional disc of Z_i. We orient D_i so that $\partial D_i \subset \partial U_i$ is homological to L_i in U_i. As in Sect. 3.2, we provide $H_*(M, E; \mathbf{R})$ with the basis

$$d_1 = [D_1, \partial D_1], ..., d_m = [D_m, \partial D_m], z_1 = [\mathrm{Z}_1, \partial \mathrm{Z}_1], ..., z_m = [\mathrm{Z}_m, \partial \mathrm{Z}_m].$$

Note that the product orientation in Z_i used to define z_i is opposite to the orientation induced from the one in M.

We provide $H_*(E; \mathbf{R})$ with the basis $([pt], t_1, ..., t_m, g_1, ..., g_{m-1})$ as in Sect. 3.1. Finally, we provide $H_*(M; \mathbf{R})$ with a basis $[pt], h, h^*, [M]$ where h is a basis

in $H_1(M;\mathbf{R})$ and h^* is the Poincaré dual basis in $H_2(M;\mathbf{R})$. The choosen bases determine the orientations ω_L, ω_M, $\omega_{(M,E)}$ in $H_*(E;\mathbf{R})$, $H_*(M;\mathbf{R})$, $H_*(M,E;\mathbf{R})$, respectively. Let $\mathcal{H}$ be the exact homological sequence of the pair (M,E) as in Sect. 3.2. Consider its torsion with respect to the choosen bases $\tau(\mathcal{H}) \in \mathbf{R}\backslash\{0\}$. Let $\tau_0 = \operatorname{sign}\tau(\mathcal{H}) = \pm 1$. In the notation of Sect. 3.2 we have $\tilde{\omega}_L = \tau_0\,\omega_M$. Then

$$\omega_L^M = (-1)^{mb_3(M)+(b_1(E)+1)(b_1(M)+m)}\,\tilde{\omega}_L = (-1)^{(m+1)b_1(M)+m}\,\tau_0\,\omega_M. \qquad (2.c)$$

We compute τ_0. The inclusion homomorphism $H_2(E;\mathbf{R}) \to H_2(M;\mathbf{R})$ is zero and therefore $\mathcal{H}$ splits as a concatenation of three acyclic chain complexes

$$H_0(E;\mathbf{R}) \to H_0(M;\mathbf{R}), \quad H_3(M;\mathbf{R}) \to H_3(M,E;\mathbf{R}) \to H_2(E;\mathbf{R}),$$

$$H_2(M;\mathbf{R}) \to H_2(M,E;\mathbf{R}) \to H_1(E;\mathbf{R}) \to H_1(M;\mathbf{R}).$$

Therefore $\tau_0 = \varepsilon_1\varepsilon_2\varepsilon_3$ where $\varepsilon_1,\varepsilon_2,\varepsilon_3$ are the signs of the torsions of these three chain complexes, respectively. The inclusion isomorphism $H_0(E;\mathbf{R}) \to H_0(M;\mathbf{R})$ is given by the unit (1×1)-matrix and hence $\varepsilon_1 = +1$. The inclusion homomorphism $H_3(M;\mathbf{R}) \to H_3(M,E;\mathbf{R})$ maps $[M]$ to $-(z_1+...+z_m)$ and the boundary homomorphism $H_3(M,E;\mathbf{R}) \to H_2(E;\mathbf{R})$ maps $z_1,...,z_m$ to $-q_1,...,-q_{m-1},q_1+...+q_{m-1}$, respectively. Now it is easy to compute that $\varepsilon_2 = -1$. Using the basis $d_1,...,d_m$ we can identify $H_2(M,E;\mathbf{R})$ with $\mathbf{R}^m$. We identify $H_1(E;\mathbf{R})$ with $(\mathbf{R}^m)^* = \operatorname{Hom}(\mathbf{R}^m,\mathbf{R})$ so that the basis $t_1,...,t_m$ is dual to $d_1,...,d_m$. With respect to these bases, the boundary homomorphism $\partial : H_2(M,E;\mathbf{R}) \to H_1(E;\mathbf{R})$ is presented by the matrix $B = [lk(L_i,L_j)]_{i,j=1}^m$ and therefore equals $\operatorname{ad}(b)$ where b is the symmetric bilinear form on $\mathbf{R}^m$ determined by B. This allows us to identify $H_2(M;\mathbf{R}) = \operatorname{Ker}\partial, H_1(M;\mathbf{R}) = \operatorname{Coker}\partial$ with $\operatorname{Ann}(b), (\operatorname{Ann}(b))^*$, respectively. Under our choice of orientations, the integral intersection index of D_i with t_j is equal to $-\delta_i^j$ where δ_i^j is the Kronecker symbol. Therefore the intersection pairing between $H_2(M;\mathbf{R})$ and $H_1(M;\mathbf{R})$ is equal to -1 times the Kronecker pairing between $\operatorname{Ann}(b)$ and $(\operatorname{Ann}(b))^*$. By formula (2.a), we have

$$\varepsilon_3 = (-1)^{b_1(M)}(-1)^{(m+1)b_1(M)}\det{}_0(B) = (-1)^{mb_1(M)}\det{}_0(B).$$

Hence $\tau_0 = -(-1)^{mb_1(M)}\det_0(B)$. Substituting this in (2.c) we obtain (2.b).

Corollaries. We can rewrite Theorems 9.2, 10.2, 11.2 replacing ω_L^M with ω_M and simultaneously inserting on the right-hand side the factor $(-1)^{b_1(M)+m+1}\det_0(B)$.

Appendix 3. Corrections and additions to [Tu5], [Tu6]

Corrections and additions to [Tu5]. Let M be a compact connected oriented 3-manifold whose boundary is either void or consists of tori. Fix a homology orientation ω of M. In the case $b_1(M) = 1$ we fix also an element $t \in H = H_1(M)$ whose projection to $H/\operatorname{Tors} H = \mathbf{Z}$ is a generator. Following [Tu5], we define a

numerical *torsion function* T_ω on the set of Euler structures on M. In the case $b_1(M) = 1$ this function depends on $t(\mathrm{mod}\,\mathrm{Tors}\,H) \in H/\mathrm{Tors}\,H$ and is denoted $T_{\omega,t}$. Let $e \in \mathrm{vect}(M)$. If $b_1(M) \geq 2$ then in the notation of Sect. 12.1, $T_\omega(e) = (\tau(M, e, \omega))_1 \in \mathbf{Z}$. If $b_1(M) = 0$ then $T_\omega(e) = (\tau(M, e, \omega))_1 \in \mathbf{Q}$. If $b_1(M) = 1$ then $T_{\omega,t}(e) = (\tau(M, e, \omega))_1^t \in \mathbf{Z}$. Clearly, $T_{-\omega} = -T_\omega$.

The torsion function T in [Tu5] coincides with T_ω under the following two restrictions on ω, t. If $\partial M = \emptyset$ then $\omega = \omega_M$ should be the canonical homology orientation of M (see Appendix 2). If $\partial M \neq \emptyset$ and $b_1(M) = 1$ then $t = t(\omega)$ should be chosen so that ω is determined by the basis $([pt], t)$ in homology.

Several inaccuracies occured in [Tu5] in the case of closed M with $b_1(M) = 1$. Their source is a mistake in the duality formulas in [Tu5, Sect. 2.7, 3.4]. In these formulas instead of $c(e)(= e/e^{-1})$ should be $(c(e))^{-1}$. As a result, in a few formulas the sign should be inverted. We give here the correct formulas. For $e \in \mathrm{vect}(M)$ denote by $K_t(e)$ the unique integer K such that $c(e) \in t^K\,\mathrm{Tors}\,H$. Set $\Sigma = \Sigma_{h \in \mathrm{Tors}\,H}\, h \in \mathbf{Z}[H]$. In Theorem 4.2.3 of [Tu5] we should have

$$\tau_t(M, e) = \tau(M, e, \omega_M) + \frac{K_t(e) + 2}{2}(1 - t)^{-1}\Sigma - (1 - t)^{-2}\Sigma \in \mathbf{Z}[H]. \qquad (3.a)$$

On p. 690, line 11 from below should be written $\tau_{t^{-1}}(M, e) = \tau_t(M, e) - (K_t(e)/2)$. On p. 691, line 18 from above should be written $r = (-K - 2)/2$. On p. 694, lines 3 and 6 should be $T_t(e) = q_t^e(1) - K_t(e)/2$ and $T_{t^{-1}}(e) = T_t(e) + K_t(e)/2$ (where the torsion functions $T_t, T_{t^{-1}}$ correspond to $\omega = \omega_M$).

We state here the duality property for T_ω. Let $e \in \mathrm{vect}(M)$. If $b_1(M) \neq 1$ then

$$T_\omega(e) = (-1)^{b_0(\partial M)}\, T_\omega(e^{-1}). \qquad (3.b)$$

If $b_1(M) = 1$ then

$$T_{\omega,t}(e) = (-1)^{b_0(\partial M)}\, T_{\omega,t^{-1}}(e^{-1}). \qquad (3.c)$$

These equalities result from the formula, proven below,

$$\overline{\tau(M, e, \omega)} = (-1)^{b_0(\partial M)}\, \tau(M, e^{-1}, \omega) \qquad (3.d)$$

where the overbar denotes the involution in $Q(H)$ sending any $h \in H$ to h^{-1}. It is clear that (3.d) implies (3.b) for $b_1(M) \neq 1$. Let us check (3.c). If $\partial M = \emptyset$ then replacing if necessary ω by $-\omega$ we can assume that $\omega = \omega_M$. By (3.a), (3.d),

$$\tau(M, e^{-1}, \omega) = \overline{\tau(M, e, \omega)} = \overline{\tau_t(M, e)} - \frac{K_t(e) + 2}{2}(1 - t^{-1})^{-1}\Sigma + (1 - t^{-1})^{-2}\Sigma.$$

Therefore

$$T_{\omega,t^{-1}}(e^{-1}) = (\overline{\tau_t(M, e)})_1 - \frac{K_t(e) + 2}{2} + 1 = (\tau_t(M, e))_1 - \frac{K_t(e)}{2} = T_{\omega,t}(e).$$

If $\partial M \neq \emptyset$ then the proof is similar using that by [Tu5], Theorem 4.2.1,

$$\tau(M, e, \omega) - (1 - t(\omega))^{-1}\Sigma \in \mathbf{Z}[H]. \qquad (3.e)$$

As an exercise, the reader can prove that if $b_1(M) = 1$ and $\partial M \neq \emptyset$ then $T_{\omega,(t(\omega))^{-1}}(e) = T_{\omega,t(\omega)}(e) - 1$.

Proof of (3.d). In the case $\partial M = \emptyset$, formula (3.d) follows from the results of [Tu4], Appendix B combined with the computation of signs in [Tu3], Appendix. We give here a proof in the case where $b_1(M) \geq 1$ and ∂M is possibly non-void. This covers all possible cases since in our setting $b_1(M) \geq 1$ whenever $\partial M \neq \emptyset$.

We first establish (3.d) when $M = E$ is the exterior of a link $L = L_1 \cup ... \cup L_m \subset S^3$. Let $k = (k_1, ..., k_m)$ be a charge on L. The key fact is the equality $\overline{\nabla}_L = (-1)^m \nabla_L$ (see Sect. 8.1). We have

$$\overline{\tau(E, e_k, \omega_L)} = \overline{\nabla(L, k)} = -t_1^{-k_1/2} ... t_m^{-k_m/2} \nabla_L(t_1^{-1/2}, ..., t_m^{-1/2})$$

$$= -(-1)^m t_1^{-k_1/2} ... t_m^{-k_m/2} \nabla_L(t_1^{1/2}, ..., t_m^{1/2}) = (-1)^m \nabla(L, -k)$$

$$= (-1)^m \tau(E, e_{-k}, \omega_L) = (-1)^m \tau(E, (e_k)^{-1}, \omega_L).$$

We claim that if (3.d) holds for a 3-manifold E with $b_1(E) \geq 2$ then it holds for a 3-manifold M with $b_1(M) \geq 1$ obtained from E by gluing m directed solid tori whose cores represent elements of infinite order, $h_1, ..., h_m \in H_1(M)$. Indeed, let $e \in \mathrm{vect}(E)$ and ω be a homology orientation of E. Let $\mathrm{in} : \mathbf{Z}[H_1(E)] \to \mathbf{Z}[H_1(M)]$ be the inclusion homomorphism. Assume first that $m = 1$ and set $h = h_1$. By Lemma 7.3.3, $\tau(M, e^M, \omega^M) = (h-1)^{-1} \mathrm{in}(\tau(E, e, \omega))$. Hence

$$\overline{\tau(M, e^M, \omega^M)} = (h^{-1} - 1)^{-1} \overline{\mathrm{in}(\tau(E, e, \omega))}$$

$$= (h^{-1} - 1)^{-1} \mathrm{in}(\overline{\tau(E, e, \omega)}) = -h(h-1)^{-1} (-1)^{b_0(\partial E)} \mathrm{in}(\tau(E, e^{-1}, \omega))$$

$$= (-1)^{b_0(\partial M)} h(h-1)^{-1} \mathrm{in}(c(e)^{-1}) \mathrm{in}(\tau(E, e, \omega))$$

$$= (-1)^{b_0(\partial M)} h \, \mathrm{in}(c(e)^{-1}) \tau(M, e^M, \omega^M)$$

$$= (-1)^{b_0(\partial M)} h \, \mathrm{in}(c(e)^{-1}) c(e^M) \tau(M, (e^M)^{-1}, \omega^M).$$

It remains to observe that $c(e^M) = \mathrm{in}(c(e)) h^{-1}$ (cf. (1.5.c)). The case $m > 1$ is similar.

Now we can accomplish the proof of (3.d) in the case $b_1(M) \geq 1$. The argument given in [Tu5], Sect. 3.9 shows that there is a framed link $L \subset M$ (with ≥ 2 components) whose components represent elements of infinite order in $H_1(M)$ and such that the exterior, E, of L in M is homeomorphic to the exterior of a link in S^3. The arguments above imply that (3.d) holds for E and therefore for M.

Corrections and additions to [Tu6]. In [Tu6], Sect. 1 the definition of the distinguished Euler structure on a directed solid torus should be the same as in the present paper. Thus, on the last line of [Tu6], Sect. 1.3 instead of $c(s_t) = t$ should be $c(s_t) = t^{-1}$. The excision formula in [Tu6], Sect. 2.3 should look like

$$\mathrm{in}_*(v(E)) = \pm \prod_{i=1}^{m} (1 - [L_i]^{-1}) \, v(M). \tag{3.f}$$

The corresponding changes should be implemented in the corollaries of this formula in [Tu6], Sect. 3. However, all the statements in [Tu6], Sect. 2 and 3 remain valid.

In [Tu6], Sect. 4.2 it is claimed that the torsion function satisfies all the axioms stated in [Tu6], Sect. 2. Here we give more details. Let $\mathcal{S}$ be the class of triples (M, ω, e) where M is a compact connected oriented 3-manifold with $b_1(M) \geq 1$ whose boundary is either void or consists of tori, ω is a homology orientation on M, and e is an Euler structure (= relative $Spin^c$-structure) on M. In the case $\partial M = \emptyset$ we assume that $\omega = \omega_M$ is induced by the orientation of M. In the case $b_1(M) = 1, \partial M = \emptyset$ the manifold M is assumed to be endowed with a homology class $t(M) \in H = H_1(M)$ whose projection to $H/\mathrm{Tors}\, H$ is a generator. We define a $\mathbf{Z}$-valued function v on $\mathcal{S}$ by

$$v(M, \omega, e) = \begin{cases} T_\omega(e^{-1}), \ if \ \ b_1(M) \geq 2, \\ T_{\omega,t}(e^{-1}), \ if \ \ b_1(M) = 1 \end{cases} \tag{3.g}$$

where $t = t(M)$ in the case $b_1(M) = 1, \partial M = \emptyset$ and $t = t(\omega)$ in the case $b_1(M) = 1, \partial M \neq \emptyset$. We claim that v satisfies the axioms stated in [Tu6], Sect. 2. By the uniqueness, v coincides with the Seiberg-Witten function $SW : \mathcal{S} \to \mathbf{Z}$ at least up to sign depending only on the underlying 3-manifold. Combining this with (3.b), (3.c) we obtain (12.1.a), (12.1.b).

Before discussing the properties of v we comment on the relations between v and τ. Assume first that $b_1(M) \geq 2$. For any $e \in \mathrm{vect}(M)$, $g \in H$,

$$(\tau(M, e, \omega))_g = (g^{-1} \tau(M, e, \omega))_1 = (\tau(M, g^{-1}e, \omega))_1$$

$$= v(M, (g^{-1}e)^{-1}, \omega) = v(M, ge^{-1}, \omega).$$

Hence $\tau(M, e, \omega) = \sum_{g \in H} v(M, ge^{-1}, \omega)\, g$. The formal expression

$$v(M) = \sum_{e \in \mathrm{vect}(M)} v(M, e, \omega)\, e \in \mathbf{Z}[\mathrm{vect}(M)]$$

can therefore be computed as follows. Fix $e_0 \in \mathrm{vect}(M)$. Then

$$v(M) = \sum_{e \in \mathrm{vect}(M)} v(M, e, \omega)\, e = \sum_{g \in H} v(M, ge_0^{-1}, \omega)\, ge_0^{-1} = \tau(M, e_0, \omega)\, e_0^{-1}.$$

Similarly, if $b_1(M) = 1$ then $(\tau(M, e, \omega))^t = \sum_{g \in H} v(M, ge^{-1}, \omega)\, g$ and

$$v(M) = \sum_{e \in \mathrm{vect}(M)} v(M, e, \omega)\, e = (\tau(M, e_0, \omega))^t\, e_0^{-1} \in \mathbf{Z}[[\mathrm{vect}(M)]]$$

where t is as in (3.g).

The function v satisfies Axioms 1 - 4 stated in [Tu6], Sect. 2. Axiom 1 (topological invariance) is obvious. Axiom 2 (first part) consists in the finiteness of

the support of v for any M with $b_1(M) \geq 2$. This follows from the inclusion $\tau(M,e,\omega) \in \mathbf{Z}[H_1(M)]$. Axiom 2 (second part) claims that the support of v is essentially t-positive for any M with $b_1(M) = 1$ where t is as in (3.g). This follows from (3.a) and (3.e). Axiom 3 amounts to the excision formula (3.f) provided $b_1(E) \geq 2$. Here in_* is an additive homomorphism of abelian groups $\mathbf{Z}[\mathrm{vect}(E)] \to \mathbf{Z}[\mathrm{vect}(M)]$ sending any $e \in \mathrm{vect}(E)$ to $e^M \in \mathrm{vect}(M)$ in the notation of Sect. 1.5. We check (3.f). Assume first that $b_1(M) \geq 2$. Fix a homology orientation ω in E and $e_0 \in \mathrm{vect}(E)$. Note that $\pm v(E)$ does not depend on the choice of ω. Let in denote the inclusion homomorphism $\mathbf{Z}[H_1(E)] \to \mathbf{Z}[H_1(M)]$. By Lemma 7.3.3,

$$\pm \mathrm{in}_*(v(E)) = \pm \mathrm{in}_*(\tau(M,e_0,\omega)\, e_0^{-1}) = \pm \mathrm{in}(\tau(M,e_0,\omega))(e_0^{-1})^M$$

$$= \pm \prod_{i=1}^{m}([L_i]-1)\, \tau(M, e_0^M, \omega^M) \prod_{i=1}^{m}[L_i]^{-1}(e_0^M)^{-1} = \pm \prod_{i=1}^{m}(1-[L_i]^{-1})\, v(M).$$

In the case $b_1(M) = 1$ the proof is similar using that

$$\prod_{i=1}^{m}([L_i]-1)\, \tau(M, e_0^M, \omega^M) = \prod_{i=1}^{m}([L_i]-1)\, (\tau(M, e_0^M, \omega^M))^t. \qquad (3.h)$$

Formula (3.h) follows from (3.a) (resp. from (3.e)) if $\partial M = \emptyset$ and $m \geq 2$ (resp. if $\partial M \neq \emptyset$). If $b_1(M) = 1, \partial M = \emptyset$ and $m = 1$ then the assumption $b_1(E) \geq 2$ implies that $[L_1] \in \mathrm{Tors}\, H_1(M)$. Then $[L_1] - 1$ annihilates Σ so that (3.h) also follows from (3.a). Finally, Axiom 4 follows from the fact that for the exteriors of links in S^3, the torsion τ coincides with the Milnor torsion [Mi1] which is equivalent to the Alexander polynomial.

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